

HELICOIDAL MINIMAL SURFACES OF PRESCRIBED GENUS, II

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Abstract: In this paper we prove that for each positive integer g , there exists a complete minimal surface of genus g that is properly embedded in three-dimensional euclidean space and that is asymptotic to the helicoid.

1. INTRODUCTION

In this paper we prove

Theorem 1. *For each positive integer g , there exists a complete minimal surface of genus g that is properly embedded in \mathbb{R}^3 and asymptotic to the helicoid.*

Let $\mathbb{S}^2(r)$ be the round sphere of radius r . Helicoidal minimal surfaces in $\mathbb{S}^2(r) \times \mathbb{R}$ of prescribed genus have been constructed by the authors in [4]. We obtain helicoidal minimal surfaces in \mathbb{R}^3 of prescribed genus by letting the radius r go to infinity.

Our model for $\mathbb{S}^2(r)$ is $\mathbb{C} \cup \{\infty\}$ with the conformal metric obtained by stereographic projection:

$$(1) \quad \lambda^2 |dz|^2 \quad \text{with } \lambda = \frac{2r^2}{r^2 + |z|^2},$$

In this model, the equator is the circle $|z| = r$. Our model for $\mathbb{S}^2(r) \times \mathbb{R}$ is $(\mathbb{C} \cup \{\infty\}) \times \mathbb{R}$ with the metric

$$(2) \quad \lambda^2 |dz|^2 + dt^2, \quad (z, t) \in (\mathbb{C} \cup \{\infty\}) \times \mathbb{R}.$$

When $r \rightarrow \infty$, this metric converges to the euclidean metric $4|dz|^2 + dt^2$ on $\mathbb{C} \times \mathbb{R} = \mathbb{R}^3$. (This metric is isometric to the standard euclidean metric by the map $(z, t) \mapsto (2z, t)$.)

Let H be the standard helicoid in \mathbb{R}^3 , defined by the equation

$$x_2 \cos x_3 = x_1 \sin x_3.$$

It turns out that H is minimal for the metric (2) for any value of r , although not complete anymore (see Section 2 in [4]). We complete it by adding the vertical line $\{\infty\} \times \mathbb{R}$, and still denote it H . This is a complete, genus zero, minimal surface in $\mathbb{S}^2 \times \mathbb{R}$.

The helicoidal minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ constructed by the authors in [4] have any prescribed genus. In this paper we only consider those of even genus, which

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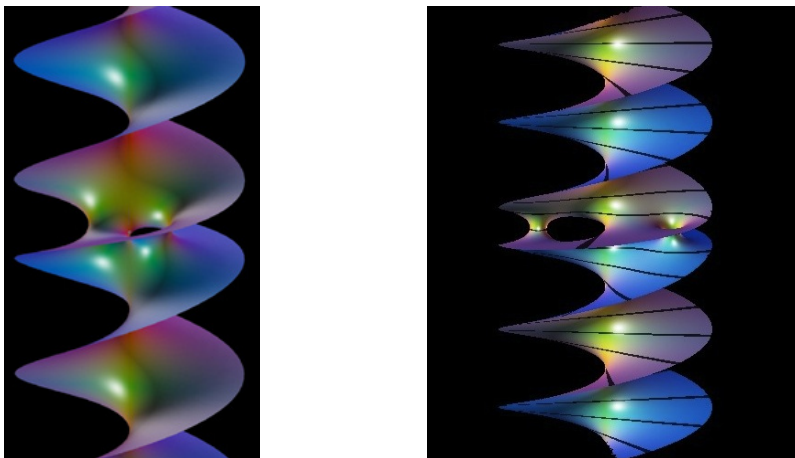


FIGURE 1. **Left:** A genus-one helicoid, computed by David Hoffman, Hermann Karcher and Fusheng Wei. **Right:** A genus-two helicoid, computed by Martin Traizet. Both surfaces were computed numerically using the Weierstrass representation and the images were made with Jim Hoffman using visualization software he helped to develop.

have one additional symmetry (denoted μ_E below). Fix some positive integer g . For any radius $r > 0$, there exist, by Theorem 1 in [4], two distinct helicoidal minimal surfaces of even genus $2g$ in $\mathbb{S}^2(r) \times \mathbb{R}$, which we denote $M_+(r)$ and $M_-(r)$. Each one has two ends (corresponding to the two ends of $\mathbb{S}^2(r) \times \mathbb{R}$), each asymptotic to H or to a vertical translate of H .

Regarding the limit as $r \rightarrow \infty$, the following result was already proved in [4], Theorem 2.

Theorem 2. *Let $s \in \{+, -\}$. Let R_n be a sequence of radii diverging to infinity. A subsequence of the genus- $2g$ surfaces $M_s(R_n)$ converges to a minimal surface M_s in \mathbb{R}^3 asymptotic to the helicoid H . The convergence is smooth convergence on compact sets. Moreover,*

- *the genus of M_s is at most g ,*
- *the genus of M_+ is even,*
- *the genus of M_- is odd,*
- *the number of points in $M_s \cap Y$ is $2 \text{ genus}(M_s) + 1$.*

The main result of this paper is the following theorem, from which Theorem 1 follows.

Theorem 3. *If g is even, then M_+ has genus g . If g is odd, then M_- has genus g .*

1.1. Pictures. A genus-2 helicoid was computed numerically by the second author in 1993 while he was a postdoc in Amherst (see Figure 1, right). Helicoids of genus up to six have been computed by Markus Schmies [6] using the theoretical techniques developed by Alexander Bobenko [2]. These surfaces were computed using

the Weierstrass Representation and the Period Problem was solved numerically. There is of course no evidence that these numerically computed examples are the same as the ones obtained in Theorem 1, but they share the same symmetries. Pictures of these numerical examples suggest that an even genus helicoid looks like a helicoid with an even number of handles far from the axis, and an odd genus helicoid looks like a genus-one helicoid with an even number of handles far from the axis, the spacing between the handles getting larger and larger as the genus increases.

2. PRELIMINARIES

2.1. Symmetries. Let us recall some notation from [4]. The real and imaginary axes in \mathbb{C} are denoted X and Y . The circle $|z| = r$ is denoted E (the letter E stands for “equator”). Note that X, Y and E are geodesics for the metric (1). We identify \mathbb{S}^2 with $\mathbb{S}^2 \times \{0\}$, so X, Y and E are horizontal geodesics in $\mathbb{S}^2 \times \mathbb{R}$. The antipodal points $(0, 0)$ and $(\infty, 0)$ are denoted O and O^* respectively. The vertical axes through O and O^* in $\mathbb{S}^2 \times \mathbb{R}$ are denoted Z and Z^* , respectively. If γ is a horizontal or vertical geodesic in $\mathbb{S}^2 \times \mathbb{R}$, the 180° rotation around γ is denoted ρ_γ . This is an isometry of $\mathbb{S}^2 \times \mathbb{R}$. The reflection in the vertical cylinder $E \times \mathbb{R}$ is denoted μ_E . This is an isometry of $\mathbb{S}^2 \times \mathbb{R}$. In our model,

$$\mu_E(z, t) = \left(\frac{r^2}{\bar{z}}, t \right).$$

The helicoid H in $\mathbb{S}^2 \times \mathbb{R}$ contains the geodesic X , the axes Z and Z^* and meets the geodesic Y orthogonally at the points O and O^* . It is invariant by $\rho_X, \rho_Z, \rho_{Z^*}, \mu_E$ (which reverse its orientation) and ρ_Y (which preserves it).

The genus- $2g$ minimal surfaces $M_+(r)$ and $M_-(r)$ in $\mathbb{S}^2(r) \times \mathbb{R}$ have the following properties (see Theorem 1 of [4]):

Proposition 1. *Let $s \in \{+, -\}$. Then:*

- (1) $M_s(r)$ is complete, properly embedded and has a top end and a bottom end, each asymptotic to H or a vertical translate of H ,
- (2) $M_s(r) \cap H = X \cup Z \cup Z^*$. In particular, $M_s(r)$ is invariant by ρ_X, ρ_Z and ρ_{Z^*} , each of which reverses its orientation.
- (3) $M_s(r)$ is invariant by the reflection μ_E , which reverses its orientation,
- (4) $M_s(r)$ meets the geodesic Y orthogonally at $4g + 2$ points and is invariant under ρ_Y , which preserves its orientation. Moreover, $(\rho_Y)_*$ acts on $H_1(M_s(r), \mathbb{Z})$ by multiplication by -1 .

2.2. Setup. Let R_n be a diverging sequence of radii. By Theorem 2, a subsequence of $M_s(R_n)$ (still denoted the same) converges to a helicoidal minimal surface M_s that is helicoidal at infinity. Let g' be the genus of M_s . By the last point of Theorem 2, $M_s \cap Y$ has exactly $2g' + 1$ points. It follows that $2g' + 1$ points of $M_s(R_n) \cap Y$ stay at bounded distance from the origin O . By μ_E -symmetry, $2g' + 1$ points of $M_s(R_n) \cap Y$ stay at bounded distance from the antipodal point O^* . There remains $4(g - g')$ points in $M_s(R_n) \cap Y$ whose distance to O and O^* is unbounded. Let

$$N = g - g'.$$

We shall prove

Theorem 4. *In the above setup, $N \leq 1$.*

Theorem 3 is a straightforward consequence of this theorem: Indeed if g is even and $s = +$, we know by Theorem 2 that g' is even so $N = 0$ and $g = g'$. If g is odd and $s = -$, then g' is odd so again $N = 0$.

Remark 1. *If we let $M_s(r) \subset \mathbb{S}^2(r) \times \mathbb{R}$, $s \in \{+, -\}$ be the two helicoidal minimal surfaces of odd genus $2g + 1$ constructed in [4] (instead of even genus), then $M_+(R_n)$ will converge subsequentially to a minimal surface M_+ of even genus g' and $\mu_E(M_+(R_n)) = M_-(R_n)$ will converge subsequentially to a minimal surface M_- of odd genus g'' . Then there are $4g + 2 - 2g' - 2g'' = 4N$ points on $M_+(R_n) \cap Y$ whose distance to O and O^* is unbounded. Following our line of argument one should be able to prove that $N \leq 1$. This, however, does not determine g' nor g'' , so is not enough to get the existence of helicoidal minimal surfaces in \mathbb{R}^3 of prescribed genus. This is the main reason why we only consider minimal surfaces of even genus in $\mathbb{S}^2 \times \mathbb{R}$. The μ_E symmetry that these surfaces possess does make the proof somewhat simpler. The only point where we use it in a fundamental way is in the proof of Proposition 11 where we use Alexandrov reflection. Another argument would be required at this point in the odd-genus case.*

To prove Theorem 4, assume that $N \geq 1$. We want to prove that $N = 1$ by studying the $4N$ points whose distance to O and O^* is unbounded. To do this, it is necessary to work on a different scale. Let R_n be a sequence of radii with $R_n \rightarrow \infty$. Define

$$M_n = \frac{1}{R_n} M_s(R_n) \subset \mathbb{S}^2(1) \times \mathbb{R}.$$

This is a minimal surface in $\mathbb{S}^2(1) \times \mathbb{R}$. Each end of M_n is asymptotic to a vertical translate of a helicoid of pitch

$$t_n = \frac{2\pi}{R_n}.$$

(The pitch of a helicoid with counterclockwise rotation is twice the distance between consecutive sheets. The standard helicoid has pitch 2π .) Observe that $t_n \rightarrow 0$. By the definition of N , the intersection $M_n \cap Y$ has $4N$ points whose distance to O and O^* is $\gg t_n$. Because M_n is symmetric with respect to 180° rotation ρ_X around X , there are $2N$ points on the positive Y -axis. We order these by increasing imaginary part:

$$p'_{1,n}, p''_{1,n}, p'_{2,n}, p''_{2,n}, \dots, p'_{N,n}, p''_{N,n}.$$

Because of the ρ_X -symmetry, the $2N$ points on the negative Y -axis are the conjugates of these points. Define $p_{j,n}$ to be the midpoint of the interval $[p'_{j,n}, p''_{j,n}]$ and $r_{j,n}$ to be half the distance in the spherical metric from $p'_{j,n}$ to $p''_{j,n}$. We have

$$0 < \operatorname{Im} p_{1,n} < \operatorname{Im} p_{2,n} < \dots < \operatorname{Im} p_{N,n}.$$

By μ_E -symmetry, which corresponds to inversion in the unit circle,

$$(3) \quad p_{N+1-i,n} = \frac{1}{\overline{p_{i,n}}}.$$

In particular, in case N is odd, $p_{\frac{N+1}{2},n} = i$.

For $\lambda > 1$ sufficiently large, let $\mathcal{Z}_n(\lambda)$ be the part of M_n lying inside of the vertical cylinders of radius λt_n around Z and Z^* :

$$(4) \quad \mathcal{Z}_n(\lambda) = \{q = (z, t) \in M_n : d(Z \cup Z^*, q) < \lambda t_n\}.$$

Also define $D_{j,n}(\lambda) = \{z : d(z, p_{j,n}) < \lambda r_{j,n}\}$. Consider the intersection of M_n with the vertical cylinder over $D_{j,n}(\lambda)$, and let $C_{j,n}(\lambda)$ denote the component of this intersection that contains the points $\{p'_{j,n}, p''_{j,n}\}$. Define

$$(5) \quad \mathcal{C}_n(\lambda) = \bigcup_{j=1}^N C_{j,n}(\lambda) \cup \overline{C_{j,n}(\lambda)}.$$

The following proposition is key to setting up the analysis we will do in this paper to show that at most one handle is lost in taking the limit as $R_n \rightarrow \infty$. In broad terms, it says that near the points $p_{j,n}$, catenoidal necks are forming on a small scale, and after removing these necks and a neighborhood of the axes, what is left is a pair of symmetric surfaces which are vertical graphs over a half-helicoid.

Proposition 2. *Let $N = g - g'$, t_n , and $M_n \subset \mathbb{S}^2(1) \times \mathbb{R}$, be as above. Then*

- i. *For each j , $1 \leq j \leq N$, the surface $\frac{1}{r_{j,n}}(M_n - p_{j,n})$ converges to the standard catenoid \mathbf{C} with vertical axis and waist circle of radius 1 in \mathbb{R}^3 . In particular, the distance (in the spherical metric) $d(p_{j,n}, p_{j+1,n})$ is $\gg r_{j,n}$. Moreover, $t_n \gg r_{j,n}$ and the $C_{j,n}(\lambda)$ are close to catenoidal necks with collapsing radii.*
- ii. *Given $\epsilon > 0$, there exists a $\lambda > 0$ such that*

$$M'_n = M_n \setminus (\mathcal{Z}_n(\lambda) \cup \mathcal{C}_n(\lambda))$$

has the following properties

- (a) *The slope of the tangent plane at any point of M'_n is less than ϵ .*
- (b) *M'_n consists of two components related by the symmetry ρ_Y , rotation by 180° around Y .*
- (c) *M'_n intersects $t_n H$ in a subset of the axis X and nowhere else, with one of its components intersecting in a ray of the positive X -axis, the other in a ray of X^- . Each component is graphical over its projection onto the half-helicoid (a component of $t_n H \setminus (Z \cup Z^*)$) that it intersects.*

This proposition is proved in Theorem 16.9 and Corollary 16.13 of [4]. The notations in [4] are slightly different: See Remark 2 below.

Passing to a subsequence, $p_j = \lim p_{j,n} \in i\mathbb{R}^+ \cup \{\infty\}$ exists for all $j \in [1, N]$. We have $p_1 \in [0, i]$, and we will consider the following three cases:

- (6)
 - Case 1: $p_1 \in (0, i)$,
 - Case 2: $p_1 = 0$,
 - Case 3: $p_1 = i$.

We will see that Case 1 and Case 2 are impossible, and that $N = 1$ in Case 3.

2.3. The physics behind the proof of Theorem 4. Theorem 4 is proved by evaluating the surface tension in the Y -direction on each catenoidal neck. Mathematically speaking, this means the flux of the horizontal Killing field tangent to the Y -circle in $\mathbb{S}^2 \times \mathbb{R}$. On one hand, this flux vanishes at each neck by ρ_Y -symmetry (see Lemma 1). On the other hand, we can compute the limit F_i of the surface tension on the i -th catenoidal neck (corresponding to $p_i = \lim p_{i,n}$) as $n \rightarrow \infty$, after suitable scaling.

Assume for simplicity that the points O, p_1, \dots, p_N and O^* are distinct. Recall that the points p_1, \dots, p_N are on the positive imaginary Y -axis. For $1 \leq j \leq N$, let $p_j = iy_j$, with $0 < y_j < \infty$. Then we will compute that

$$F_i = c_i^2 \frac{1 - y_i^2}{1 + y_i^2} + \sum_{\substack{j=1 \\ j \neq i}}^N c_i c_j f(y_i, y_j)$$

where the numbers c_i are positive and proportional to the size of the catenoidal necks and

$$f(x, y) = \frac{-2\pi^2}{(\log x - \log y)|\log x - \log y + i\pi|^2}.$$

Observe that f is antisymmetric and $f(x, y) > 0$ when $0 < x < y$. We can think of the point p_i as a particle with mass c_i and interpret F_i as a force of gravitation type. The particles p_1, \dots, p_N are attracted to each other and we can interpret the first term by saying that each particle p_i is repelled from the fixed antipodal points O and O^* . All forces F_i must vanish. It is physically clear that no equilibrium is possible unless $N = 1$ and $p_1 = i$. Indeed in any other case, $F_1 > 0$.

This strategy is similar to the one followed in [7] and [8]. The main technical difficulty is that we cannot guarantee that the points O, p_1, \dots, p_N and O^* are distinct. The distinction between Cases 1, 2 and 3 in (6) stems from this problem.

2.4. The space $\widetilde{\mathbb{C}^*}$. To compute forces we need to express M_n as a graph. For this, we need to express the helicoid itself as a graph, away from its axes Z and Z^* . Let \mathbb{C}^* be the universal cover of \mathbb{C}^* . Of course, one can identify $\widetilde{\mathbb{C}^*}$ with \mathbb{C} by mean of the exponential function. It will be more convenient to see $\widetilde{\mathbb{C}^*}$ as the covering space obtained by analytical continuation of $\log z$, so each point of $\widetilde{\mathbb{C}^*}$ is a point of \mathbb{C}^* together with a determination of its argument : points are couples $(z, \arg(z))$, although in general we just write z . The following two involutions of \mathbb{C}^* will be of interest:

- $(z, \arg(z)) \mapsto (\bar{z}, -\arg(z))$, which we write simply as $z \mapsto \bar{z}$. The fixed points are $\arg z = 0$.
- $(z, \arg(z)) \mapsto (1/\bar{z}, \arg(z))$, which we write simply as $z \mapsto 1/\bar{z}$. The fixed points are $|z| = 1$.

The graph of the function $\frac{t}{2\pi} \arg z$ on $\widetilde{\mathbb{C}^*}$ is one half of a helicoid of pitch t .

2.5. The domain Ω_n and the functions f_n and u_n . By Proposition 2, away from the axes $Z \cup Z^*$ and the points $p_{j,n}$, we may consider M_n to be the union of two multigraphs. We wish to express this part of M_n as a pair of graphs over a subdomain of $\widetilde{\mathbb{C}^*}$. We will allow ourselves the freedom to write z for a point $(z, \arg z) \in \widetilde{\mathbb{C}^*}$ when its argument is clear from the context. Thus we will write $p_{j,n}$ for the point $(p_{j,n}, \pi/2)$ in $\widetilde{\mathbb{C}^*}$ corresponding to the points on $M_n \cap Y$ in Proposition 2. Define

$$(7) \quad D_n(\lambda) = \{ (z, \arg z) : |z| < \lambda t_n \text{ or } |z| > \frac{1}{\lambda t_n} \},$$

$$(8) \quad D_{j,n}(\lambda) = \{ (z, \arg z) : d(p_{j,n}, z) < \lambda r_{j,n} \text{ and } 0 < \arg z < \pi \}$$

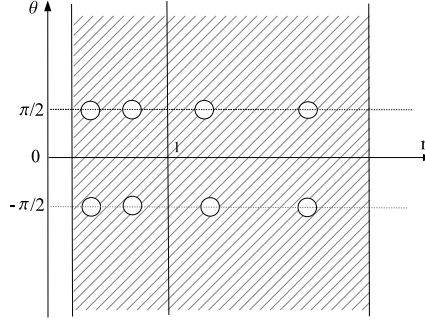


FIGURE 2. The domain Ω_n in polar coordinates, $z = re^{i\theta}$. The function u_n is positive for $\theta > 0$. The line $r = 1$ corresponds to the unit circle $|z| = 1$. The white strip on the left corresponds to the projection of the vertical cylinder of radius λt_n about the Z -axis, and the region to the right of the shaded domain is its image by the inversion through the unit circle. The small disks correspond to the vertical cylinders of radius $\lambda r_{j,n}$ (in the spherical metric).

and

$$(9) \quad \Omega_n = \Omega_n(\lambda) = \widetilde{\mathbb{C}}^* \setminus \left(D_n(\lambda) \cup \bigcup_{j=1}^N D_{j,n}(\lambda) \cup \overline{D_{j,n}(\lambda)} \right).$$

According to Statement *ii.* of Proposition 2, there exists a $\lambda > 0$ such that for sufficiently large n ,

$$M'_n = M_n \cap (\Omega_n(\lambda) \times \mathbb{R})$$

is the union of two graphs related by ρ_Y -symmetry, and each graph intersects the helicoid of pitch t_n in a subset of the X -axis. Only one of these graphs can contain points on the positive X -axis. We choose this component and write it as the graph of a function f_n on the domain Ω_n . We may write

$$(10) \quad f_n(z) = \frac{t_n}{2\pi} \arg z - u_n(z).$$

The function u_n has the following properties:

- $$(11) \quad \begin{aligned} & \bullet \quad u_n(\bar{z}) = -u_n(z). \text{ In particular, } u_n = 0 \text{ on } \arg z = 0. \\ & \bullet \quad u_n(1/\bar{z}) = u_n(z) \text{ In particular, } \partial u_n / \partial \nu = 0 \text{ on } |z| = 1. \\ & \bullet \quad 0 < u_n < t_n/2 \text{ when } \arg z > 0. \end{aligned}$$

The first two assertions follow from the symmetries of M_n . See Proposition 1 (Statements 2 and 3), and the discussion preceding it. The third assertion follows Proposition 2, Statement *ii.c.*, which implies that

$$0 < |u_n| < t_n/2$$

when $\arg z > 0$, since the vertical distance between the sheets of $t_n H$ is equal to $t_n/2$. Now choose a point z_0 in the domain of f_n that is near the a point $p_{j,n}$. Then $|f_n(z_0)|$ is small, and $\arg z_0$ is near $\pi/2$. Hence $f_n(z_0) \sim t_n/4 - u_n(z_0)$, which implies that $u_n(z_0) > 0$. We conclude that $0 < u_n < t_n/2$ when $\arg z > 0$, as claimed.

Remark 2. *There are some notational differences between [4] and the present paper:*

- In [4], z denotes the third coordinate in \mathbb{R}^3 . Here $z = x + iy$ is a complex variable.
- In [4], the pitch of a helicoid is denoted 2η . Here it is denoted t .
- In Section 16 of [4], the angle θ is measured from the positive Y -axis, whereas here, it is measured from the positive X -axis.
- The conformal factor for the spherical metric in [4] is slightly different from (1). It is chosen there so that it converges to 1 as $r \rightarrow \infty$.
- In paper [4], $M_s(r)$ has genus g , whereas here it has genus $2g$.

2.6. Organization of the paper. We deal with Cases 1, 2 and 3, as listed in (6), separately. In each case, we first state, without proof, a proposition which describes the asymptotic behavior of the function u_n defined by (10) as $n \rightarrow \infty$. We use this result to compute forces and obtain the required result (namely, $N = 1$ or a contradiction). Then, we prove the proposition. Finally, an Appendix contains analytic and geometric results relevant to minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$, which are used in this paper.

3. CASE 1: $p_1 \in (0, i)$

For $p \in \widetilde{\mathbb{C}}^*$, let h_p be the harmonic function defined on $\widetilde{\mathbb{C}}^* \setminus \{p, \bar{p}\}$ by

$$(12) \quad h_p(z) = -\log \left| \frac{\log z - \log p}{\log z - \log \bar{p}} \right|.$$

Note that since p and z are in $\widetilde{\mathbb{C}}^*$, both come with a determination of their logarithm, so the function h_p is well defined. This function has the same symmetries as u_n :

$$(13) \quad \begin{aligned} & \bullet \quad h_p(\bar{z}) = -h_p(z), \\ & \bullet \quad h_p(1/\bar{z}) = h_{1/\bar{p}}(z). \\ & \bullet \quad \text{Moreover, if } \arg p \text{ and } \arg z \text{ are positive then } h_p(z) > 0. \end{aligned}$$

Remark 3. *The function $(z, p) \mapsto -h_p(z)$ is the Green function of the domain $\arg z > 0$ of $\widetilde{\mathbb{C}}^*$.*

Recall that $p_i = \lim p_{i,n}$. It might happen that several points p_j are equal to p_i . In this case, we say that we have a cluster at p_i . Let m be the number of distinct points amongst p_1, \dots, p_N . Relabel the points so that p_1, \dots, p_m are distinct and

$$\operatorname{Im} p_1 < \operatorname{Im} p_2 < \dots < \operatorname{Im} p_m.$$

(In other words, we have selected one point in each cluster.) Let us define

$$(14) \quad \tilde{u}_n = \frac{|\log t_n|}{t_n} u_n.$$

Proposition 3. *Assume that $p_1 \neq 0$. Then there exists a subsequence and non-negative real numbers c_0, \dots, c_m such that*

$$(15) \quad \tilde{u}(z) := \lim \tilde{u}_n(z) = c_0 \arg z + \sum_{i=1}^m c_i h_{p_i}(z).$$

The convergence is the usual smooth uniform convergence on compact subsets of $\widetilde{\mathbb{C}}^*$ minus the points $p_i, -p_i$ for $1 \leq i \leq m$. Moreover, for $1 \leq i \leq m$,

$$(16) \quad c_i = \lim_{t_n} \frac{|\log t_n|}{t_n} \frac{\phi_{i,n}}{2\pi}$$

where $\phi_{i,n}$ is the vertical flux of M_n on the graph of f_n restricted to the circle $C(p_i, \varepsilon)$ for a fixed, small enough ε .

In other words, $\phi_{i,n}$ is the sum of the vertical fluxes on the catenoidal necks corresponding to the points $p_{j,n}$ such that $p_j = p_i$.

Remark 4. We allow $p_1 = i$ as this proposition will be used in Case 3, Section 5.

This proposition is proved in Section 3.2 by estimating the Laplacian of u_n and constructing an explicit barrier, from which we deduce that a subsequence converges to a limit harmonic function on $\widetilde{\mathbb{C}}^*$ with logarithmic singularities at $\pm p_1, \dots, \pm p_m$.

Remark 5. In Proposition 3, it is easy to show using Harnack's inequality that we can choose numbers $\lambda_n > 0$ so that $\lambda_n u_n$ converges subsequentially to a nonzero limit of the form (15). (One fixes a point z_0 and lets $\lambda_n = 1/u_n(z_0)$.) However, for us it is crucial that we can choose λ_n to be $\frac{|\log t_n|}{t_n}$; it means that in later calculations, we will be able to ignore terms that are $o(\frac{|\log t_n|}{t_n})$.

For all we know at this point, the limit \tilde{u} might be zero. We will prove this is not the case:

Proposition 4. For each $i \in [1, m]$, $c_i > 0$.

This proposition is proved in Section 3.3 using a height estimate, Proposition 19, to estimate the vertical flux of the catenoidal necks.

From now on assume that $p_1 \in (0, i)$. Fix some small number ε and let F_n be the flux of the Killing field χ_Y on the circle $C(p_1, \varepsilon)$. The field χ_Y is the Killing field associated with rotations with respect to poles whose equator is the Y -circle (see Proposition 17 in Appendix A.3). On one hand, we have:

Lemma 1. $F_n = 0$.

Proof. Let C_n be the graph of f_n restricted to the circle $C(p_1, \varepsilon)$. By Proposition 1, statement (4), C_n together with its image $\rho_Y(C_n)$ bound a compact region in M_n . Thus the flux of the Killing field χ_Y on $C_n \cup \rho_Y(C_n)$ is 0. By ρ_Y -symmetry, this flux is twice the flux F_n of χ_Y on C_n . Thus $F_n = 0$. \square

On the other hand, F_n can be computed using Proposition 18 from Appendix A.3:

$$\begin{aligned} F_n &= -\operatorname{Im} \int_{C(p_1, \varepsilon)} 2 \left(\frac{\partial}{\partial z} \left(\frac{t_n}{2\pi} \arg z - u_n \right) \right)^2 \frac{i}{2} (1 - z^2) dz + O(t_n^4) \\ &= -\operatorname{Re} \int_{C(p_1, \varepsilon)} \left(\frac{t_n}{4\pi i z} - u_{n,z} \right)^2 (1 - z^2) dz + O(t_n^4) \\ &= -\operatorname{Re} \int_{C(p_1, \varepsilon)} \left(\frac{-t_n^2}{16\pi^2 z^2} - \frac{2t_n}{4\pi i z} u_{n,z} + (u_{n,z})^2 \right) (1 - z^2) dz + O(t_n^4) \\ (17) \quad &= \operatorname{Re} \int_{C(p_1, \varepsilon)} \left(\frac{2t_n}{4\pi i z} u_{n,z} - (u_{n,z})^2 \right) (1 - z^2) dz + O(t_n^4) \end{aligned}$$

The second equation comes from $\frac{\partial}{\partial z} \arg z = \frac{1}{2iz}$. The fourth equation is a consequence of the fact that $\frac{1-z^2}{z^2}$ has no residue at $p_1 \neq 0$. The first term in (17) (the cross-product) is a priori the leading term. However we can prove that this term can be neglected:

Proposition 5.

$$(18) \quad \lim \left(\frac{\log t_n}{t_n} \right)^2 F_n = -\operatorname{Re} \int_{C(p_1, \varepsilon)} (\tilde{u}_z)^2 (1 - z^2) dz$$

where \tilde{u} is defined in (15) as the limit of $\frac{|\log t_n|}{t_n} u_n$.

This proposition is proved in Section 3.4 using a Laurent series expansion to estimate the first term in (17).

Assuming these results, we now prove

Proposition 6. *Case 1 is impossible.*

Proof: According to Lemma 1, the flux F_n is zero. Hence the limit in (18) is zero. We compute that limit and show that it is nonzero.

Differentiating equation (15), we get

$$\tilde{u}_z = \frac{c_0}{2iz} - \sum_{i=1}^m \frac{c_i}{2z} \left(\frac{1}{\log z - \log p_i} - \frac{1}{\log z - \log \bar{p}_i} \right).$$

Therefore,

$$\begin{aligned} & \operatorname{Res}_{p_1} (\tilde{u}_z)^2 (1 - z^2) \\ &= \operatorname{Res}_{p_1} \frac{1 - z^2}{4z^2} \left[\frac{c_1^2}{(\log z - \log p_1)^2} + 2 \frac{c_1}{\log z - \log p_1} \right. \\ & \quad \left. \left(\frac{-c_0}{i} - \frac{c_1}{\log z - \log \bar{p}_1} + \sum_{i=2}^m \frac{c_i}{\log z - \log p_i} - \frac{c_i}{\log z - \log \bar{p}_i} \right) \right] \\ &= -\frac{c_1^2(1 + p_1^2)}{4p_1} + \frac{c_1(1 - p_1^2)}{2p_1} \left(\frac{-c_0}{i} - \frac{c_1}{\log p_1 - \log \bar{p}_1} \right. \\ & \quad \left. + \sum_{i=2}^m \frac{c_i}{\log p_1 - \log p_i} - \frac{c_i}{\log p_1 - \log \bar{p}_i} \right). \end{aligned}$$

(See Proposition 23 in Appendix A.6 for the residue computations.) Write $p_j = iy_j$ for $1 \leq j \leq m$ so all y_j are positive numbers. By Lemma 1, equation (18) and the Residue Theorem,

$$\begin{aligned} 0 &= -\operatorname{Re} \int_{C(p_1, \varepsilon)} (\tilde{u}_z)^2 (1 - z^2) dz \\ &= -\operatorname{Re} \left[2\pi i \frac{y_1^2 + 1}{4iy_1} \left(c_1^2 \frac{y_1^2 - 1}{y_1^2 + 1} + 2c_1 \left(-\frac{c_0}{i} - \frac{c_1}{i\pi} \right. \right. \right. \\ & \quad \left. \left. + \sum_{i=2}^m \frac{c_i}{\log y_1 - \log y_i} - \frac{c_i}{\log y_1 - \log y_i + i\pi} \right) \right) \right] \\ (19) \quad &= \frac{\pi(y_1^2 + 1)}{2y_1} \left[c_1^2 \frac{1 - y_1^2}{y_1^2 + 1} + \sum_{i=2}^m \frac{-2\pi^2 c_1 c_i}{(\log y_1 - \log y_i) |\log y_1 - \log y_i + i\pi|^2} \right]. \end{aligned}$$

Now $y_1 < 1$ and $y_1 < y_i$ for all $i \geq 2$, so all terms in (19) are positive. This contradiction proves Proposition 6. \square

Remark 6. *The bracketed term in (19) is precisely the expression for the force F_1 in Section 2.3.*

3.1. Barriers. In this section we introduce various barriers that will be used to prove Proposition 3. Fix some $\alpha \in (0, 1)$.

Definition 1. A_n is the set of points $(z, \arg z)$ in $\widetilde{\mathbb{C}}^*$ which satisfy $t_n^\alpha < |z| < 1$ and $\arg z > 0$, minus the disks $D(p_{i,n}, t_n^\alpha)$ for $1 \leq i \leq N$.

It is clear that $A_n \subset \Omega_n$ for large n , since $t_n^\alpha \gg t_n$. Moreover, if $z \in A_n$ then $d(z, \partial\Omega_n) \geq t_n^\alpha/2$.

Remark 7. *We work in the hemisphere $|z| \leq 1$ where the conformal factor of the spherical metric in (1) satisfies $1 \leq \lambda \leq 2$. Hence euclidean and spherical distances are comparable. We will use euclidean distance. Also the euclidean and spherical Laplacians are comparable. The symbol Δ will mean euclidean Laplacian.*

By the disk $D(p, r)$ in $\widetilde{\mathbb{C}}^$ (for small r) we mean the points $(z, \arg z)$ such that $|z - p| < r$ and $\arg z$ is close to $\arg p$.*

Let δ be the function on A_n defined by

$$\delta(z) = \begin{cases} \min\{|z|, |z - p_{1,n}|, \dots, |z - p_{N,n}|\} & \text{if } 0 < \arg z < \pi \\ |z| & \text{if } \arg(z) \geq \pi \end{cases}$$

Lemma 2. *There exists a constant C_1 such that in the domain A_n , the function u_n satisfies*

$$|\Delta u_n| \leq C_1 \frac{t_n^3}{\delta^4}$$

Proof: the function $f_n(z) = \frac{t_n}{2\pi} \arg z - u_n(z)$ satisfies the minimal surface equation, and $|\Delta f_n| = |\Delta u_n|$. The proposition then follows from Proposition 15 in Appendix A.1 (a straightforward application of Schauder estimate). More precisely:

- if $0 < \arg z < \pi$, we apply Proposition 15 on the domain

$$A'_n = \{w \in \Omega_n : -\pi/2 < \arg w < 3\pi/2, |w| < 2\}.$$

The distance $d(z, \partial A'_n)$ is comparable to $\delta(z)$. The function f_n is bounded by $3t_n/4$.

- If $k\pi \leq \arg z < k\pi + \pi$ for some $k \geq 1$, we apply Proposition 15 to the function $f_n - \frac{k}{2}t_n$ and the domain

$$A'_n = \{w \in \Omega_n : k\pi - \pi/2 < \arg w < k\pi + 3\pi/2, |w| < 2\}.$$

The distance $d(z, \partial A'_n)$ is comparable to $|z|$. The function $f_n - \frac{k}{2}t_n$ is again bounded by $3t_n/4$. \square

Next, we need to construct a function whose Laplacian is greater than $1/\delta^4$, in order to compensate for the fact that u_n is not quite harmonic. Let $\chi : \mathbb{R}^+ \rightarrow [0, 1]$ be a fixed, smooth function such that $\chi \equiv 1$ on $[0, \pi]$ and $\chi \equiv 0$ on $[2\pi, \infty)$.

Lemma 3. *There exists a constant $C_2 \geq 1$ such that the function g_n defined on A_n by*

$$g_n(z) = \frac{C_2}{|z|^2} + \chi(\arg z) \sum_{i=1}^N \frac{1}{|z - p_{i,n}|^2}$$

satisfies

$$(20) \quad \Delta g_n \geq \frac{4}{\delta^4}.$$

Moreover $\partial g_n / \partial \nu \leq 0$ on $|z| = 1$ and

$$(21) \quad g_n \leq \frac{C_2 + N}{t_n^{2\alpha}} \quad \text{in } A_n.$$

Proof: The inequality (21) follows immediately from the definitions of g_n and A_n . The function f defined in polar coordinate by $f(r, \theta) = 1/r^2$ satisfies

$$|\nabla f| = \frac{2}{r^3}, \quad \Delta f = \frac{4}{r^4}.$$

Hence for $\arg z \geq 2\pi$, (20) is satisfied for any $C_2 \geq 1$. Suppose $0 < \arg z < \pi$. Then

$$\Delta g_n = \frac{4C_2}{|z|^4} + \sum_{i=1}^N \frac{4}{|z - p_{i,n}|^4} \geq \frac{4}{\delta^4}$$

so again, (20) is satisfied for any $C_2 \geq 1$. If $\theta = \arg z \in [\pi, 2\pi]$, we have $|z - p_{i,n}| \geq |z| = r$ and

$$|\nabla \chi(\arg z)| \leq \frac{C}{r}, \quad |\Delta \chi(\arg z)| \leq \frac{C}{r^2}.$$

Hence

$$\left| \Delta \frac{\chi(\arg z)}{|z - p_{i,n}|^2} \right| \leq \frac{C}{r^2} \frac{1}{r^2} + 2 \frac{C}{r} \frac{2}{r^3} + \frac{4}{r^4}.$$

Therefore, $\Delta g_n \geq 4/r^4$ provided C_2 is large enough. (The constant C_2 only depends on N and a bound on χ' and χ'' .) This completes the proof of (20). \square

We need a harmonic function on $\widetilde{\mathbb{C}}^*$ that is greater than $|\log t|$ on $|z| = t$. A good candidate would be $-\log |z|$. However this function has the wrong Neumann data on the unit circle. We propose the following:

Lemma 4. *For $0 < t < 1$, the harmonic function $H_t(z)$ defined for $z \in \widetilde{\mathbb{C}}^*$, $\arg z > 0$ by*

$$H_t(z) = \operatorname{Im} \left(\frac{\log t \log z}{\log t + i \log z} \right)$$

has the following properties :

- (1) $H_t(z) > 0$ if $\arg z > 0$,
- (2) $H_t(1/\bar{z}) = H_t(z)$, hence $\partial H_t / \partial \nu = 0$ on $|z| = 1$,
- (3) $H_t(z) \geq |\log t|/2$ if $|z| = t$,
- (4) for fixed t , $H_t(z) \geq |\log t|/2$ when $\arg z \rightarrow \infty$, uniformly with respect to $|z|$ in $t \leq |z| \leq 1$,
- (5) for fixed z , $H_t(z) \rightarrow \arg z$ when $t \rightarrow 0$,
- (6) $H_t(z) \leq |\log z|$ if $\arg z > 0$.

Proof : it suffices to compute $H_t(z)$ in polar coordinates $z = re^{i\theta}$:

$$H_t(z) = \frac{(\log t)^2 \theta + |\log t|((\log r)^2 + \theta^2)}{(\log t - \theta)^2 + (\log r)^2}.$$

The first two points follow. If $r = t$ then

$$H_t(z) = \frac{|\log t|}{2} \left(1 + \frac{\theta^2}{2(\log t)^2 + 2|\log t|\theta + \theta^2} \right) \geq \frac{|\log t|}{2}$$

which proves point 3. If $t \leq r \leq 1$ then

$$H_t(z) \geq \frac{(\log t)^2 \theta + |\log t|\theta^2}{(\log t - \theta)^2 + (\log t)^2}$$

which gives point 4. Point 5 is elementary. For the last point, write

$$\left| \frac{\log t \log z}{\log t + i \log z} \right| \leq \left| \frac{\log t \log z}{\log t} \right| = |\log z|.$$

□

3.2. Proof of Proposition 3. The function \tilde{u}_n defined in (14) has the following properties in A_n :

$$(22) \quad \begin{aligned} & \bullet \quad |\Delta \tilde{u}_n| \leq C_1 \frac{t_n^2 |\log t_n|}{\delta^4} \text{ by Lemma 2,} \\ & \bullet \quad \tilde{u}_n \leq |\log t_n|/2, \\ & \bullet \quad \tilde{u}_n = 0 \text{ on } \arg z = 0, \\ & \bullet \quad \partial \tilde{u}_n / \partial \nu = 0 \text{ on } |z| = 1. \end{aligned}$$

The last three properties follow from (11) and the fact that $A_n \subset \Omega_n$. Consider the barrier $v_n = v_{1,n} + v_{2,n} + v_{3,n}$ where

$$\begin{aligned} v_{1,n}(z) &= -C_1 t_n^2 |\log t_n| g_n(z) + C_1 (C_2 + N) t_n^{2-2\alpha} |\log t_n|, \\ v_{2,n}(z) &= \frac{1}{\alpha} \sum_{i=1}^N h_{p_{i,n}}(z) = -\frac{1}{\alpha} \sum_{i=1}^N \log \left| \frac{\log z - \log p_{i,n}}{\log z - \log \bar{p}_{i,n}} \right|, \\ v_{3,n}(z) &= \frac{1}{\alpha} H_{t_n^\alpha}(z). \end{aligned}$$

The function $v_{1,n}$ is positive in A_n by the estimate (21) of Lemma 3. Observe that the second term in the expression for $v_{1,n}$ tends to 0 as $n \rightarrow \infty$ since $\alpha < 1$. The functions $v_{2,n}$ and $v_{3,n}$ are harmonic and positive in A_n (see point (1) of Lemma 4 for $v_{3,n}$).

By (13) and the symmetry of the set $\{p_{1,n}, \dots, p_{N,n}\}$ (see (3)), the function $v_{2,n}$ satisfies $v_{2,n}(1/\bar{z}) = v_{2,n}(z)$. Hence $\partial v_{2,n} / \partial \nu = 0$ on the unit circle. By point (2) of Lemma 4, $\partial v_{3,n} / \partial \nu = 0$ on the unit circle. Therefore by Lemma 3,

$$\frac{\partial v_n}{\partial \nu} = \frac{\partial v_{1,n}}{\partial \nu} \geq 0 \text{ on } |z| = 1.$$

Because $p_{i,n} \rightarrow p_i \neq 0$, we have on the circle $C(p_{i,n}, t_n^\alpha)$

$$\log |\log z - \log p_{i,n}| \simeq \log |z - p_{i,n}|$$

Hence for large n and for $1 \leq i \leq N$

$$v_{2,n} \geq \frac{1}{2\alpha} |\log t_n^\alpha| \geq \tilde{u}_n \quad \text{on } C(p_{i,n}, t_n^\alpha).$$

Using point (3) of Lemma 4 and the second statement of (22), we have $v_{3,n} \geq \tilde{u}_n$ on the boundary component $|z| = t_n^\alpha$. So we have

- $\Delta \tilde{u}_n \geq \Delta v_n$ in A_n ,
- $\tilde{u}_n \leq v_n$ on the boundaries $\arg z = 0$, $|z| = t_n^\alpha$ and $C(p_{i,n}, t_n^\alpha)$,
- $\partial \tilde{u}_n / \partial \nu \leq \partial v_n / \partial \nu$ on the boundary $|z| = 1$,
- $\tilde{u}_n \leq v_n$ when $\arg z \rightarrow \infty$.

(The first statement follows from (20) and the first statement of (22).)

By the maximum principle, we have $\tilde{u}_n \leq v_n$ in A_n .

For any compact set K of the set $\{z \in \mathbb{C}^* : |z| \leq 1, \arg z \geq 0\} \setminus \{p_1, \dots, p_m\}$, the function v_n is bounded by $C(K)$ on K . (For $v_{3,n}$, use the last point of Lemma 4.) Then by symmetry, u_n is bounded by $C(K)$ on $K \cup \bar{K} \cup \sigma(K) \cup \sigma(\bar{K})$, where σ denotes the inversion $z \mapsto \bar{z}$. Let

$$\Omega_\infty = \lim_{n \rightarrow \infty} \Omega_n = \mathbb{C}^* \setminus \{\pm p_1, \dots, \pm p_m\}.$$

Then \tilde{u}_n is bounded on compact subsets of Ω_∞ . By standard PDE theory, passing to a subsequence, \tilde{u}_n has a limit \tilde{u} . The convergence is the uniform smooth convergence on compact subsets of Ω_∞ . The limit has the following properties

- \tilde{u} is harmonic in Ω_∞ . This follows from the first point of (22).
- $\tilde{u}(\bar{z}) = -\tilde{u}(z)$ and $\tilde{u}(1/\bar{z}) = \tilde{u}(z)$.
- $\tilde{u}(z) \geq 0$ if $\arg z \geq 0$.

Note that either $\tilde{u} \equiv 0$ or \tilde{u} is positive in $\arg z > 0$. Using the fact that $\log : \mathbb{C}^* \rightarrow \mathbb{C}$ is biholomorphic, the following lemma tells us that \tilde{u} has the form given by equation (15).

Lemma 5. *Let H be the upper half plane $\operatorname{Im} z > 0$ in \mathbb{C} . Let u be a positive harmonic function in $H \setminus \{q_1, \dots, q_m\}$ with boundary value $u = 0$ on \mathbb{R} . Then there exists non-negative constants c_0, \dots, c_m such that*

$$u(z) = c_0 \operatorname{Im} z - \sum_{i=1}^m c_i \log \left| \frac{z - q_i}{z - \bar{q}_i} \right|.$$

This lemma easily follows from the following two facts and the maximum principle:

- If u is a positive harmonic function in the punctured disk $D(p, \varepsilon) \setminus \{p\}$ then $u(z) = -c \log |z - p| + v(z)$ where v is harmonic in the disk. (Bôcher Theorem, [1], Theorem 3.9)
- If u is a positive harmonic function in H with boundary value 0 on \mathbb{R} then $u = c \operatorname{Im} z$. ([1], Theorem 7.22).

To conclude the proof of Proposition 3, it remains to compute the numbers c_i for $1 \leq i \leq m$. Recall that $\phi_{i,n}$ is the vertical flux of M_n on the graph of f_n restricted to the circle $C(p_i, \varepsilon)$. By Proposition 18,

$$\phi_{i,n} = \operatorname{Im} \int_{C(p_i, \varepsilon)} (2f_{n,z} + O(t_n^2)) dz = \operatorname{Im} \int_{C(p_i, \varepsilon)} (-2u_{n,z} + O(t_n^2)) dz.$$

Now

$$\lim_{t_n} \frac{|\log t_n|}{t_n} u_n = -c_i \log |z - p_i| + \text{harmonic} \quad \text{near } p_i.$$

$$\lim_{t_n} \frac{|\log t_n|}{t_n} 2u_{n,z} = -\frac{c_i}{z - p_i} + \text{holomorphic} \quad \text{near } p_i.$$

Hence by the Residue Theorem,

$$\lim_{t_n} \frac{|\log t_n|}{t_n} \phi_{i,n} = 2\pi c_i.$$

This finishes the proof of Proposition 3. \square

As a corollary of the proof of Proposition 3, we have an estimate of u_n that we will need in Section 5.5. For convenience, we state it here as a lemma.

Fix some $\beta \in (0, \alpha)$ and let $A'_n \subset A_n$ be the domain defined as A_n in Definition 1, replacing α by β , namely: A'_n is the set of points $(z, \arg z)$ in $\widetilde{\mathbb{C}}^*$ which satisfy $t_n^\beta < |z| < 1$ and $\arg z > 0$, minus the disks $D(p_{i,n}, t_n^\beta)$ for $1 \leq i \leq N$.

Lemma 6. *Assume that $p_1 \neq 0$. Then for n large enough (depending only on β and a lower bound on $|p_1|$), we have*

$$u_n \leq (N+2) \frac{\beta}{\alpha} t_n \quad \text{in } A'_n.$$

Recalling that $u_n < t_n/2$, this lemma is useful when β is small. We will use it to get information about the level sets of u_n .

Proof: as we have seen in the Proof of Proposition 3, we have in A_n

$$(24) \quad u_n \leq \frac{t_n}{|\log t_n|} v_n = \frac{t_n}{|\log t_n|} (v_{1,n} + v_{2,n} + v_{3,n}).$$

We need to estimate the functions $v_{1,n}$, $v_{2,n}$ and $v_{3,n}$ in A'_n . We have in A_n

$$v_{1,n} \leq C_1(C_2 + N)t_n^{2-2\alpha} |\log t_n| = o(|\log t_n|).$$

By point 6 of Lemma 4, we have in A'_n

$$v_{3,n} \leq \frac{1}{\alpha} |\log z| \leq \frac{1}{\alpha} |\log t_n^\beta| = \frac{\beta}{\alpha} |\log t_n|.$$

Regarding the function $v_{2,n}$, we need to estimate each function $h_{p_{i,n}}$ in the domain A'_n . The function $h_{p_{i,n}}$ is harmonic in the domain

$$\{z \in \widetilde{\mathbb{C}}^* : \arg z > 0, t_n^\beta < |z| < 1\} \setminus D(p_{i,n}, t_n^\beta)$$

and goes to 0 as $\arg z \rightarrow \infty$, so its maximum is on the boundary. Since $h_{p_{i,n}}(1/\bar{z}) = h_{p_{i,n}}(z)$, the maximum is not on the circle $|z| = 1$ (because it would be an interior maximum of $h_{p_{i,n}}$). Also $h_{p_{i,n}} = 0$ on $\arg z = 0$. Therefore, the maximum is either on $|z| = t_n^\beta$ or on the circle $C(p_{i,n}, t_n^\beta)$. On $|z| = t_n^\beta$, we have $h_{p_{i,n}} \rightarrow 0$ because $p_{i,n}$ is bounded away from 0. On the circle $C(p_{i,n}, t_n^\beta)$, we have for n large

$$\log z - \log p_{i,n} \simeq \frac{1}{p_{i,n}} (z - p_{i,n})$$

$$|\log z - \log p_{i,n}| \geq \frac{t_n^\beta}{2|p_{i,n}|}.$$

Hence

$$-\log |\log z - \log p_{i,n}| \leq \log(2|p_{i,n}|) + \beta |\log t_n|.$$

Also,

$$\log |\log z - \log \overline{p_{i,n}}| \leq \log(|\log z| + |\log p_{i,n}|) \simeq \log(2|\log p_{i,n}|).$$

Since $|p_{i,n}|$ is bounded away from 0, this gives for n large enough

$$h_{p_{i,n}} \leq C + \beta |\log t_n| \quad \text{in } A'_n.$$

Hence

$$v_{2,n} \leq C + N \frac{\beta}{\alpha} |\log t_n|.$$

Collecting all terms, we get, for n large enough:

$$v_n \leq C + (N+1) \frac{\beta}{\alpha} |\log t_n| \leq (N+2) \frac{\beta}{\alpha} |\log t_n| \quad \text{in } A'_n.$$

Using (24), the lemma follows. \square

3.3. Proof of Proposition 4. We continue with the notation of the end of the previous section. Fix some index i and let $J = \{j \in [1, N] : p_j = p_i\}$. Passing to a subsequence, we may assume that

$$r_{i,n} = \max\{r_{j,n} : j \in J\}.$$

(The numbers $r_{j,n}$ have been defined in Section 2.2.) Fix some positive ε such that $|p_j - p_i| \geq 2\varepsilon$ for $j \notin J$.

From Statement *i.* of Proposition 2, we know that near $p_{j,n}$ the surface M_n is close to a vertical catenoid with waist circle of radius $r_{j,n}$. More precisely, $\frac{1}{r_{j,n}}(M_n - p_{i,n})$ converges to the standard catenoid

$$x_3 = \cosh^{-1} \sqrt{x_1^2 + x_2^2}.$$

Since the vertical flux of the standard catenoid is 2π , we have

$$(25) \quad \phi_{i,n} \simeq 2\pi \sum_{j \in J} r_{j,n} \leq 2\pi |J| r_{i,n}.$$

Let

$$h_{j,n} = r_{j,n} \cosh^{-1}(2\lambda).$$

Observe that $h_{j,n} \ll t_n$. Consider the intersection of M_n with the plane at height $h_{j,n}$ and project it on the horizontal plane. There is one component which is close to the circle $C(p_{j,n}, 2\lambda r_{j,n})$. We call this component $\gamma_{j,n}$. Observe that $\gamma_{j,n} \subset \Omega_n$ and $f_n = h_{j,n}$ on $\gamma_{j,n}$. Let $D_{j,n}$ be the disk bounded by $\gamma_{j,n}$.

We now estimate f_n on the circle $C(p_{i,n}, \varepsilon)$. By Proposition 3, we know that $|u_n| = O(\frac{t_n}{|\log t_n|})$. Hence $f_n = \frac{t_n}{2\pi} \arg z - u_n(z) \sim \frac{t_n}{2\pi} \arg z$ on $C(p_{i,n}, \varepsilon)$. Since $p_{i,n}$ is on the positive imaginary axis, $\arg z = \pi/2 + O(\varepsilon)$ on $C(p_{i,n}, \varepsilon)$. Hence $f_n(z) \sim \frac{t_n}{4}$ on $C(p_{i,n}, \varepsilon)$. Consequently, the level set $f_n = \frac{t_n}{8}$ inside $\Omega_n \cap D(p_{i,n}, \varepsilon)$ is a closed curve, possibly with several components. We select the component which encloses the point $p_{i,n}$ and call it Γ_n . (Note that by choosing a very slightly different height, we may assume that Γ_n is a regular curve). Let D_n be the disk bounded by Γ_n . Let

$$\Omega'_n = D_n \setminus \bigcup_{j \in J} D_{j,n}$$

Then $\Omega'_n \subset \Omega_n$. We are now able to apply the height estimate of Appendix A.4. We apply Proposition 19 with $r_1 = \lambda r_{i,n}$, $r_2 = \varepsilon$, $h = t_n/8 - h_{i,n} \simeq t_n/8$ and f equal to the function $f_n(z - p_{i,n}) - t_n/8$. (Observe that by Proposition 2, Statement *ii.*,

we may assume that $|\nabla f_n| \leq 1$. Also the fact that $\partial f_n / \partial \nu < 0$ on $\gamma_{j,n}$ follows from the convergence to a catenoid.) We obtain

$$\frac{t_n}{8} - h_{i,n} \leq \frac{\sqrt{2}}{\pi} \phi_{i,n} \log \frac{\varepsilon}{\lambda r_{i,n}}.$$

Using (25), this gives for n large enough

$$(26) \quad \frac{t_n}{9} \leq \frac{\sqrt{2}}{\pi} \phi_{i,n} \log \frac{2\pi|J|\varepsilon}{\lambda \phi_{i,n}}$$

This implies

$$(27) \quad \phi_{i,n} \geq \frac{2\pi|J|\varepsilon}{\lambda} t_n^2$$

for n large. To see this, suppose that $\phi_{i,n} < \frac{2\pi|J|\varepsilon}{\lambda} t_n^2$. Substituting in (26), we get

$$C_1 t_n \leq t_n^2 |\log t_n|$$

for some constant $C_1 > 0$. This is clearly a contradiction since $t_n |\log t_n| \rightarrow 0$. Substitution of (27) in (26) gives

$$\frac{t_n}{9} \leq \frac{2\sqrt{2}}{\pi} \phi_{i,n} |\log t_n|$$

which implies that $\frac{|\log t_n|}{t_n} \phi_{i,n}$ is bounded below by a positive constant independent of n . Therefore, the coefficient c_i defined in (16) is positive, as desired. \square

Remark 8. Together with (25), this gives

$$(28) \quad r_{i,n} \geq \frac{1}{36|J|\sqrt{2}|\log t_n|} \frac{t_n}{t_n}$$

for large n . This is a lower bound on the size of the largest catenoidal neck in the cluster corresponding to p_i . We have no lower bound for $r_{j,n}$ if $j \in J$, $j \neq i$. Conceptually, we could have $r_{j,n} = o(\frac{t_n}{|\log t_n|})$, although this seems unlikely.

3.4. Proof of Proposition 5. Let $g_n = u_{n,z}$. We have to prove

$$\lim \left(\frac{\log t_n}{t_n} \right)^2 \operatorname{Re} \int_{C(p_1, \varepsilon)} \frac{2t_n}{4\pi i z} u_{n,z} (1-z)^2 dz = 0,$$

i.e., that

$$\operatorname{Re} \int_{C(p_1, \varepsilon)} \frac{1}{2iz} g_n(z) (1-z^2) dz = o \left(\frac{t_n}{(\log t_n)^2} \right).$$

Fix some α such that $0 < \alpha < \frac{1}{2}$ and some small $\varepsilon > 0$. Let J be the set of indices such that $p_j = p_1$. Consider the domain

$$A_n = D(p_1, \varepsilon) - \bigcup_{j \in J} D(p_{j,n}, t_n^\alpha) \subset \Omega_n.$$

By Proposition 15 in Appendix A.1, we have in A_n

$$|g_{n,\bar{z}}| = \frac{1}{4} |\Delta u_n| = \frac{1}{4} |\Delta f_n| \leq C t_n^{3-4\alpha}.$$

$$|\nabla f_n| \leq C t_n^{1-\alpha}.$$

As the gradient of $t_n \arg z$ is $O(t_n)$ in A_n , this gives

$$|\nabla u_n| \leq C t_n^{1-\alpha}.$$

Hence

$$(29) \quad |g_n| \leq Ct_n^{1-\alpha}.$$

Proposition 21 gives us the formula

$$g_n(z) = g^+(z) + \sum_{j \in J} g_j^-(z) + \frac{1}{2\pi i} \int_{A_n} \frac{g_{n,\bar{z}}(w)}{w-z} dw \wedge \overline{dw}$$

where of course the functions g^+ and g_j^- depend on n .

- The function g^+ is holomorphic in $D(p_1, \varepsilon)$ so does not contribute to the integral.
- The last term is bounded by $Ct_n^{3-4\alpha}$. (The integral of $dw \wedge \overline{dw}/(w-z)$ is uniformly convergent.) Therefore we need $3-4\alpha > 1$, namely $\alpha < \frac{1}{2}$ so that the contribution of this term to the integral is $o(t_n/(\log t_n)^2)$.

Remark 9. *This is a crude estimate. The laplacian Δu_n is bounded by Ct_n^3/d^4 , where d is distance to the boundary. Integrating this estimate one get that this term is less than $Ct_n^{3-2\alpha}$, which is better. But one still needs $\alpha < 1$ to ensure that this term is $o(t_n/(\log t_n)^2)$.*

- Each function f_j^- is expanded in series as in Proposition 21. By Proposition 22 in Appendix A.5, each residue $a_{j,1}$ is real. Hence

$$(30) \quad \operatorname{Re} \int_{C(p_1, \varepsilon)} \frac{1}{2iz} \frac{a_{j,1}}{z - p_{j,n}} (1 - z^2) dz = a_{j,1} \operatorname{Re} \left(\frac{2\pi i}{2ip_{j,n}} (1 - p_{j,n}^2) \right) = 0$$

because $p_{j,n}$ is imaginary, so $a_{j,1}$ does not contribute to the integral.

- It remains to estimate the coefficients $a_{j,k}$ for $k \geq 2$. Using (29),

$$|a_{j,k}| = \left| \frac{1}{2\pi i} \int_{C(p_{j,n}, t_n^\alpha)} g_n(z) (z - p_{j,n})^{k-1} dz \right| \leq Ct_n^{1+(k-1)\alpha}$$

If $z \in C(p_1, \varepsilon)$, then $|z - p_{j,n}| \geq \varepsilon/2$, so

$$\left| \sum_{k=2}^{\infty} a_{j,k} (z - p_{j,n})^{-k} \right| \leq C \sum_{k \geq 2} t_n^{1+(k-1)\alpha} \left(\frac{2}{\varepsilon} \right)^k \leq \frac{4C}{\varepsilon^2} t_n^{1+\alpha} \sum_{k=2}^{\infty} \left(\frac{2t_n^\alpha}{\varepsilon} \right)^{k-2}.$$

The last sum converges because $\alpha > 0$. Hence the contribution of this term to the integral is $o(t_n/(\log t_n)^2)$ as desired. \square

4. CASE 2: $p_1 = 0$

In this case we make a blow up at the origin. Let

$$R_n = \frac{1}{|p_{1,n}|}$$

(Here we assume again that the points $p_{i,n}$ are ordered by increasing imaginary part as in Section 2.2.) Let $\widehat{M}_n = R_n M_n$. This is a helicoidal minimal surface in $\mathbb{S}^2(R_n) \times \mathbb{R}$ with pitch

$$\widehat{t}_n = R_n t_n.$$

By choice of $p_{1,n}$, we have $|p_{1,n}| \gg t_n$, so $\lim \hat{t}_n = 0$. Let $\hat{\Omega}_n = R_n \Omega_n$. \widehat{M}_n is the graph on $\hat{\Omega}_n$ of the function

$$\hat{f}_n(z) = \frac{\hat{t}_n}{2\pi} \arg z - \hat{u}_n(z)$$

where

$$\hat{u}_n(z) = R_n u_n\left(\frac{z}{R_n}\right).$$

Let $\hat{p}_{i,n} = R_n p_{i,n}$. Passing to a subsequence

$$\hat{p}_j = \lim \hat{p}_{j,n} \in [i, \infty]$$

exists for $j \in [1, N]$ and we have $\hat{p}_1 = i$. Let m be the number of distinct, finite points amongst $\hat{p}_1, \dots, \hat{p}_N$. Relabel the points so that $\hat{p}_1, \dots, \hat{p}_m$ are distinct and

$$1 = \text{Im } \hat{p}_1 < \text{Im } \hat{p}_2 < \dots < \text{Im } \hat{p}_m.$$

Proposition 7. *Passing to a subsequence,*

$$\lim \frac{|\log \hat{t}_n|}{\hat{t}_n} \hat{u}_n(z) = \hat{c}_0 \arg z + \sum_{i=1}^m \hat{c}_i h_{\hat{p}_i}(z).$$

The convergence is the smooth uniform convergence on compact subsets of $\widehat{\mathbb{C}}^$ minus the points $\pm \hat{p}_i$, for $1 \leq i \leq m$. The numbers \hat{c}_i for $1 \leq i \leq m$ are given by*

$$\hat{c}_i = \lim \frac{|\log \hat{t}_n|}{\hat{t}_n} \frac{\hat{\phi}_{i,n}}{2\pi}$$

where $\hat{\phi}_{i,n}$ is the vertical flux of \widehat{M}_n on the graph of \hat{f}_n restricted to the circle $C(\hat{p}_i, \varepsilon)$, for some fixed small enough ε . Moreover, $\hat{c}_i > 0$ for $1 \leq i \leq m$.

This proposition is proved in Section 4.1. The proof is very similar to the proof of Proposition 3, and Proposition 4 for the last statement.

Fix some small $\varepsilon > 0$. Let F_n be the flux of the Killing fields χ_Y on the circle $C(\hat{p}_1, \varepsilon)$ on \widehat{M}_n . Since we are in $\mathbb{S}^2(R_n) \times \mathbb{R}$,

$$\chi_Y = \frac{i}{2} \left(1 - \frac{z^2}{R_n^2}\right).$$

$$F_n = -\text{Im} \int_{C(\hat{p}_1, \varepsilon)} 2 \left(\frac{\partial}{\partial z} \left(\frac{\hat{t}_n}{2\pi} \arg z - \hat{u}_n \right) \right)^2 \frac{i}{2} \left(1 - \frac{z^2}{R_n^2}\right) dz + O((\hat{t}_n)^4).$$

Expand the square. As in Case 1, the cross product term can be neglected and since $R_n \rightarrow \infty$:

Proposition 8.

$$\lim \left(\frac{\log \hat{t}_n}{\hat{t}_n} \right)^2 F_n = -\lim \left(\frac{\log \hat{t}_n}{\hat{t}_n} \right)^2 \text{Re} \int_{C(\hat{p}_1, \varepsilon)} (\hat{u}_{n,z})^2 dz$$

(Same proof as Proposition 5).

Assuming these results, we now prove

Proposition 9. *Case 2 is impossible.*

Proof: Write $\hat{p}_j = iy_j$. By the same computation as in Section 3, we get (the only difference is that there is no $(1 - z^2)$ factor)

$$-\operatorname{Re} \int_{C(\hat{p}_1, \varepsilon)} (\tilde{u}_z)^2 dz = \frac{\pi}{2y_1} \left(\hat{c}_1^2 + \sum_{i=2}^m \frac{-2\pi^2 \hat{c}_1 \hat{c}_i}{(\log y_1 - \log y_i) |\log y_1 - \log y_i + i\pi|^2} \right).$$

Again, since $y_i > y_1$ for $i \geq 2$, all terms are positive, contradiction. \square

4.1. Proof of Proposition 7. The setup of Proposition 7 is the same as Proposition 3 except that we are in $\mathbb{S}^2(R_n) \times \mathbb{R}$ with $R_n \rightarrow \infty$ instead of $\mathbb{S}^2(1) \times \mathbb{R}$, and the pitch is \hat{t}_n . Remember that $\lim t_n = 0$.

From now on forget all hats: write t_n instead of \hat{t}_n , u_n instead of \hat{u}_n , $p_{i,n}$ instead of $\hat{p}_{i,n}$, etc...

The proof of Proposition 7 is substantially the same as the proofs of Propositions 3 and 4. The main difference is that the equatorial circle $|z| = 1$ becomes $|z| = R_n$.

- The definition of the domain A_n is the same with $|z| < 1$ replaced by $|z| < R_n$.
- Lemma 2 is the same (recall that now $p_{i,n}$ means $\hat{p}_{i,n}$).
- Lemma 3 is the same. The last statement must be replaced by $\partial g_n / \partial \nu \leq 0$ on $|z| = R$ for $R \geq 1$.
- Lemma 4 is the same, we do not change the definition of the function H_t . Instead of point 3, we need $\partial H_t / \partial \nu \geq 0$ on $|z| = R$ for $R \geq 1$. This is true by the following computation:

$$\frac{\partial H_t}{\partial r} = \frac{2 \log r (\log t)^2 (|\log t| + \theta)}{((\log t - \theta)^2 + (\log r)^2)^2}.$$

- The definition of the function \tilde{u}_n is the same, and it has the same properties, except that the last point must be replaced by $\partial \tilde{u}_n / \partial \nu = 0$ on $|z| = R_n$.
- The definition of the function $v_{2,n}$ is the same (with $\hat{p}_{i,n}$ in place of $p_{i,n}$), now it is symmetric with respect to the circle $|z| = R_n$.
- At the end, K is a compact of the set $\{z \in \mathbb{C}^*, \arg z \geq 0\} \setminus \{\hat{p}_1, \dots, \hat{p}_m\}$. The fact that $v_{2,n}$ is uniformly bounded on K requires some care, maybe, because some points $\hat{p}_{i,n}$ are not bounded: it is true by the fact that if $\arg z$ and $\arg p$ are positive, then

$$|\log z - \log p| \leq |\log z - \log \bar{p}|.$$

- The proof of the last point is exactly the proof of Proposition 4, working in $\mathbb{S}^2(R_n) \times \mathbb{R}$ instead of $\mathbb{S}^2(1) \times \mathbb{R}$. \square

5. CASE 3: $p_1 = i$

Note that in this case, all points $p_{j,n}$ converge to i , for $j \in [1, N]$. We distinguish two sub-cases:

- Case 3a: there exists $\beta > 0$ such that $|p_{1,n} - i| \leq t_n^\beta$ for n large enough,
- Case 3b: for all $\beta > 0$, $|p_{1,n} - i| \geq t_n^\beta$ for n large enough.

(Here we assume again that the points $p_{i,n}$ are ordered by increasing imaginary part as in Section 2.2.) Roughly speaking, in Case 3a, all points $p_{j,n}$ converge to i quickly, whereas in Case 3b, at least two ($p_{1,n}$ and $p_{N,n}$ by symmetry) converge to

i very slowly. We will see in Proposition 11 that $N = 1$ and $p_{1,n} = i$ in Case 3a, and in Proposition 14 that Case 3b is impossible.

In both cases, we make a blowup at i as follows : Let $\varphi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be the rotation of angle $\pi/2$ which fixes the Y circle and maps i to 0. Explicitly, in our model of $\mathbb{S}^2(1)$

$$\varphi(z) = \frac{z - i}{1 - iz}, \quad \varphi^{-1}(z) = \frac{z + i}{1 + iz}.$$

It exchanges the equator E and the great circle X . φ lifts in a natural way to an isometry $\widehat{\varphi}$ of $\mathbb{S}^2 \times \mathbb{R}$. We first apply the isometry $\widehat{\varphi}$ and then we scale by $1/\mu_n$ where the ratio μ_n goes to zero and will be chosen later, depending on the case. Let

$$\widehat{M}_n = \frac{1}{\mu_n} \widehat{\varphi}(M_n) \subset \mathbb{S}^2(1/\mu_n) \times \mathbb{R},$$

$$\widehat{\Omega}_n = \frac{1}{\mu_n} \varphi(\Omega), \quad \widehat{p}_{i,n} = \frac{1}{\mu_n} \varphi(p_{i,n}), \quad \widehat{t}_n = \frac{t_n}{\mu_n}.$$

The minimal surface \widehat{M}_n is the graph over $\widehat{\Omega}_n$ of the function

$$\widehat{f}_n(z) = \frac{1}{\mu_n} f_n(\varphi^{-1}(\mu_n z)) = \widehat{t}_n w_n(z) - \widehat{u}_n(z)$$

where

$$(31) \quad w_n(z) = \frac{1}{2\pi} \arg \left(\frac{\mu_n z + i}{1 + i\mu_n z} \right)$$

$$\widehat{u}_n(z) = \frac{1}{\mu_n} u_n(\varphi^{-1}(\mu_n z)).$$

5.1. Case 3a. In this case, fix some positive number α such that $\alpha < \min\{\beta, \frac{1}{8}\}$, and take $\mu_n = t_n^\alpha$. Then for all $j \in [1, N]$, $|p_{j,n} - i| = o(\mu_n)$, so $\lim \widehat{p}_{j,n} = 0$.

Proposition 10. *In Case 3a, passing to a subsequence,*

$$(32) \quad \lim_{\widehat{t}_n} \frac{|\log t_n|}{\widehat{t}_n} (\widehat{u}_n(z) - \widehat{u}_n(z_0)) = -c(\log |z| - \log |z_0|).$$

The convergence is the uniform smooth convergence on compact subsets of $\mathbb{C} \setminus \{0\}$. (Here z_0 is an arbitrary fixed nonzero complex number.) The constant c is positive.

The Proof is in Section 5.4.

Remark 10. *In fact*

$$\lim_{\widehat{t}_n} \frac{|\log t_n|}{\widehat{t}_n} \widehat{u}_n(z) = \infty$$

for all z , so it is necessary to subtract something to get a finite limit. Because of this, we believe it is not possible to prove this proposition by a barrier argument as in the proof of Proposition 3. Instead, we will prove the convergence of the derivative $\widehat{u}_{n,z}$ using the Cauchy Pompeiu integral formula for C^1 functions.

We now prove

Proposition 11. *In Case 3a, $N = 1$.*

Proof: From (31),

$$w_n(z) = \frac{1}{2\pi} \left(\frac{\pi}{2} + O(\mu_n) \right) = \frac{1}{4}(1 + O(t_n^\alpha)).$$

Since $\alpha > 0$, $t_n^\alpha \rightarrow 0$ so using Equation (32) of Proposition 10,

$$\widehat{f}_n(z) - \frac{\widehat{t}_n}{4} + \widehat{u}_n(z_0) \simeq c \frac{\widehat{t}_n}{|\log t_n|} (\log |z| - \log |z_0|).$$

From this we conclude that for n large enough, the level curves of \widehat{f}_n are convex. Back to the original scale, we have found a horizontal convex curve γ_n which encloses N catenoidal necks and is invariant under reflection in the vertical cylinder $E \times \mathbb{R}$. In particular, this curve γ_n is a graph on each side of $E \times \mathbb{R}$. Consider the domain on M_n which is bounded by γ_n and its symmetric image with respect to the Y -circle. By Alexandrov reflection (see Appendix A.2), this domain must be symmetric with respect to the vertical cylinder $E \times \mathbb{R}$ – which we already know – and must be a graph on each side of $E \times \mathbb{R}$. This implies that the centers of all necks must be on the circle E . But $E \cap Y^+$ is a single point. Hence there is only one neck: $N = 1$. \square

5.2. Case 3b. In this case we take $\mu_n = |p_{1,n} - i|$. Passing to a subsequence, the limits

$$\widehat{p}_j = \lim \widehat{p}_{j,n} \in \left[-\frac{i}{2}, \frac{i}{2} \right]$$

exist for all $j \in [1, N]$. Moreover, we have

$$\widehat{p}_1 = -\frac{i}{2} \text{ and } \widehat{p}_N = \frac{i}{2}.$$

(The $\frac{1}{2}$ comes from the fact that the rotation φ distorts euclidean lengths by the factor $\frac{1}{2}$ at i .) Let m be the number of distinct points amongst $\widehat{p}_1, \dots, \widehat{p}_N$. Observe that $m \geq 2$ because we know that \widehat{p}_1 and \widehat{p}_N are distinct. Relabel the points so that $\widehat{p}_1, \dots, \widehat{p}_m$ are distinct and

$$\text{Im } \widehat{p}_1 < \text{Im } \widehat{p}_2 < \dots < \text{Im } \widehat{p}_m.$$

Proposition 12. *In Case 3b, passing to a subsequence,*

$$\widetilde{u}(z) := \lim_{\widehat{t}_n} \frac{|\log t_n|}{\widehat{t}_n} (\widehat{u}_n(z) - \widehat{u}_n(z_0)) = \sum_{i=1}^m -\widehat{c}_i (\log |z - \widehat{p}_i| - \log |z_0 - \widehat{p}_i|).$$

The convergence is the uniform smooth convergence on compact subsets of \mathbb{C} minus the points $\widehat{p}_1, \dots, \widehat{p}_m$. (Here z_0 is an arbitrary fixed complex number different from these points.) The constants \widehat{c}_i are positive.

The proof of this proposition is in Section 5.5.

Fix some small number $\varepsilon > 0$. Let F_n be the flux of the Killing field χ_Y on the circle $C(\widehat{p}_1, \varepsilon)$ on \widehat{M}_n . Because of the scaling we are in $\mathbb{S}^2(1/\mu_n) \times \mathbb{R}$ so

$$\chi_Y(z) = \frac{i}{2}(1 - \mu_n^2 z^2).$$

Hence using Proposition 18 in Appendix A.3,

$$(33) \quad F_n = -\text{Im} \int_{C(\widehat{p}_1, \varepsilon)} 2 (\widehat{t}_n w_{n,z} - \widehat{u}_{n,z})^2 \frac{i}{2} (1 - \mu_n^2 z^2) + O((\widehat{t}_n)^4).$$

Expand the square. Then as in Case 1, the cross-product term can be neglected, so the leading term is the one involving $(\widehat{u}_{n,z})^2$ and since $\mu_n \rightarrow 0$:

Proposition 13.

$$(34) \quad \lim \left(\frac{\log t_n}{\widehat{t}_n} \right)^2 F_n = - \lim \left(\frac{\log t_n}{\widehat{t}_n} \right)^2 \operatorname{Re} \int_{C(\widehat{p}_1, \varepsilon)} (\widehat{u}_{n,z})^2 dz.$$

This proposition is proved in Section 5.6. The proof is similar to Proposition 5.

We now prove

Proposition 14. *Case 3b is impossible.*

Proof: According to Lemma 1, the flux F_n is equal to zero. Hence the left-hand side of (34) is zero. By Propositions 12 and 13,

$$(35) \quad 0 = - \operatorname{Re} \int_{C(\widehat{p}_1, \varepsilon)} (\widetilde{u}_z)^2$$

On the other hand,

$$\begin{aligned} \widetilde{u}_z &= - \sum_{i=1}^m \frac{\widehat{c}_i}{2(z - \widehat{p}_i)} \\ \operatorname{Res}_{\widehat{p}_1} (\widetilde{u}_z)^2 &= \frac{1}{2} \sum_{i=2}^m \frac{\widehat{c}_1 \widehat{c}_i}{\widehat{p}_1 - \widehat{p}_i}. \end{aligned}$$

Write $\widehat{p}_i = iy_i$, then

$$- \int_{C(\widehat{p}_1, \varepsilon)} (\widetilde{u}_z)^2 = -\pi \sum_{i=2}^m \frac{\widehat{c}_1 \widehat{c}_i}{y_1 - y_i}.$$

Since $m \geq 2$, $y_1 < y_i$ for all $i \geq 2$ and $\widehat{c}_i > 0$ for all i by Proposition 12, this is positive, contradicting (35). \square

This completes the proof of the main theorem, modulo the proof of Propositions 10, 12 and 13, which were used in the analysis of Cases 3a and 3b. We prove these propositions in Sections 5.4, 5.5 and 5.6 respectively, using an estimate that we prove in the next section.

5.3. An estimate of $\int |\nabla u_n|$. By Proposition 3, we have, since all points $p_{j,n}$ converge to i ,

$$\lim \frac{|\log t_n|}{t_n} u_n = c_0 \arg z - c_1 \log \left| \frac{\log z - \log i}{\log z + \log i} \right|.$$

Moreover, c_1 is positive by Proposition 4. The convergence is the smooth convergence on compact subsets of $\widehat{\mathbb{C}}^* \setminus \{i, -i\}$. From this we get, for fixed $\varepsilon > 0$,

$$(36) \quad \int_{C(i, \varepsilon)} |\nabla u_n| \leq C \frac{t_n}{|\log t_n|}.$$

Let $i \in [1, n]$ be the index such that $r_{i,n} = \max\{r_{j,n} : 1 \leq j \leq N\}$. Let $\phi_n = \phi_{i,n}$ be the vertical flux of M_n on the graph of f_n restricted to $C(p_{i,n}, \varepsilon)$. By the last point of Proposition 3, we have

$$\phi_n \leq C \frac{t_n}{|\log t_n|}$$

for some constant C . We use Proposition 20 with $r_1 = \lambda r_{i,n}$ and $r_2 = \varepsilon$ as in the proof of Proposition 4, and

$$r'_1 = (t_n)^{1/4}, \quad r'_2 = (t_n)^{1/8}$$

The proposition tells us that for each $j \in [1, N]$, there exists a number r , which we call $r'_{j,n}$, such that

$$(37) \quad (t_n)^{1/4} \leq r'_{j,n} \leq (t_n)^{1/8}$$

and

$$\int_{C(p_{j,n}, r'_{j,n}) \cap \Omega_n} |\nabla f_n| \leq \sqrt{8} \phi_n \left(\log \frac{\varepsilon}{\lambda r_{i,n}} \right)^{1/2} \left(\log \frac{(t_n)^{1/8}}{(t_n)^{1/4}} \right)^{-1/2}.$$

Using (28), we have

$$\log \frac{\varepsilon}{\lambda r_{i,n}} \leq \log \frac{\varepsilon |\log t_n|}{\lambda C_1 t_n} \leq C_2 |\log t_n|$$

for some positive constants C_1 and C_2 . This gives

$$\int_{C(p_{j,n}, r'_{j,n}) \cap \Omega_n} |\nabla f_n| \leq C \phi_n \leq C \frac{t_n}{|\log t_n|}.$$

Now since $|\nabla \arg z| \simeq 1$ near i ,

$$\int_{C(p_{j,n}, r'_{j,n})} t_n |\nabla \arg z| \leq C t_n^{1+1/8} = o\left(\frac{t_n}{|\log t_n|}\right).$$

Hence

$$\int_{C(p_{j,n}, r'_{j,n}) \cap \Omega_n} |\nabla u_n| \leq C \frac{t_n}{|\log t_n|}.$$

Consider the domain

$$(38) \quad U_n = D(i, \varepsilon) \setminus \bigcup_{j=1}^N \overline{D}(p_{j,n}, r'_{j,n}).$$

Since $r'_{j,n} \gg t_n \gg r_{j,n}$, we have $\overline{U}_n \subset \Omega_n$ and

$$(39) \quad d(U_n, \partial \Omega_n) \geq \frac{1}{2} (t_n)^{1/4}.$$

Also, since $\partial U_n \subset \Omega_n$,

$$\partial U_n \subset C(i, \varepsilon) \cup \bigcup_{j=1}^N (C(p_{j,n}, r'_{j,n}) \cap \Omega_n).$$

This implies

$$(40) \quad \int_{\partial U_n} |\nabla u_n| \leq C \frac{t_n}{|\log t_n|}.$$

This is the estimate we will use in the next sections.

5.4. Proof of Proposition 10 (Case 3a). Let $\beta > 0$ be the number given by the hypothesis of case 3a. Recall that we have fixed some positive number α such that $0 < \alpha < \min\{\beta, \frac{1}{8}\}$, that $\mu_n = t_n^\alpha$, $\hat{t}_n = \frac{t_n}{\mu_n}$ and $\varphi_n = \frac{1}{\mu_n} \varphi$. Let U_n be the domain defined in (38) and $\hat{U}_n = \varphi_n(U_n)$. Since $\mu_n \gg t_n^{1/8} \geq r'_{j,n}$ by (37), we have

$$\lim \hat{U}_n = \mathbb{C}^*.$$

Since φ_n is conformal, we have, using (40) (recall the definition of \hat{u}_n in (31))

$$\int_{\partial \hat{U}_n} |\nabla \hat{u}_n| = \int_{\partial \hat{U}_n} \frac{1}{\mu_n} |\nabla(u_n \circ \varphi_n^{-1})| = \frac{1}{\mu_n} \int_{\partial U_n} |\nabla u_n| \leq C \frac{t_n}{\mu_n |\log t_n|} = C \frac{\hat{t}_n}{|\log \hat{t}_n|}.$$

Using (39), we have

$$d(\widehat{U}_n, \partial\widehat{\Omega}_n) \geq \frac{(t_n)^{1/4}}{4\mu_n}.$$

By Proposition 15 in Appendix A.1 (Interior gradient and Laplacian estimate)

$$|\Delta\widehat{u}_n| = |\Delta\widehat{f}_n| \leq C \frac{(\widehat{t}_n)^3}{((t_n)^{1/4}/(4\mu_n))^4} = C\mu_n t_n^2 \quad \text{in } \widehat{U}_n.$$

Let

$$\widetilde{u}_n = \frac{|\log t_n|}{\widehat{t}_n} (\widehat{u}_n - \widehat{u}_n(z_0)).$$

Proposition 10 asserts that a subsequence of the \widetilde{u}_n converge to $-c(\log|z| - \log|z_0|)$, where c is a real positive constant. By the above estimates,

$$(41) \quad \int_{\partial\widehat{U}_n} |\nabla\widetilde{u}_n| \leq C$$

and

$$(42) \quad |\Delta\widetilde{u}_n| \leq C\mu_n^2 t_n |\log t_n| \quad \text{in } \widehat{U}_n.$$

Let K be a compact set of \mathbb{C}^* . For n large enough, K is included in \widehat{U}_n . The Cauchy Pompeiu integral formula (Equation (46) in Appendix A.5) gives for $\zeta \in K$

$$\widetilde{u}_{n,z}(\zeta) = \frac{1}{2\pi i} \int_{\partial\widehat{U}_n} \frac{\widetilde{u}_{n,z}(z)}{z - \zeta} dz + \frac{1}{8\pi i} \int_{\widehat{U}_n} \frac{\Delta\widetilde{u}_n(z)}{z - \zeta} dz \wedge \overline{dz}.$$

We estimate each integral in the obvious way, using (41) in the first line and (42) in the third line:

$$\begin{aligned} \left| \int_{\partial\widehat{U}_n} \frac{\widetilde{u}_{n,z}}{z - \zeta} \right| &\leq \frac{1}{d(\zeta, \partial\widehat{U}_n)} \int_{\partial\widehat{U}_n} |\nabla\widetilde{u}_n| \leq \frac{C}{d(\zeta, \partial\widehat{U}_n)} \rightarrow \frac{C}{|\zeta|}. \\ \int_{\widehat{U}_n} \frac{dx dy}{|z - \zeta|} &\leq \int_{D(0, \varepsilon/\mu_n)} \frac{dx dy}{|z - \zeta|} \leq 2\pi \int_{r=0}^{2\varepsilon/\mu_n} \frac{r dr}{r} = 4\pi \frac{\varepsilon}{\mu_n}. \\ \left| \int_{\widehat{U}_n} \frac{\Delta\widetilde{u}_n}{z - \zeta} dx dy \right| &\leq C\mu_n t_n |\log t_n| \rightarrow 0. \end{aligned}$$

Hence for n large enough, we have in K

$$|\widetilde{u}_{n,z}(\zeta)| \leq \frac{C}{|\zeta|}$$

for a constant C independent of K . Passing to a subsequence, $\widetilde{u}_{n,z}$ converges smoothly on compact sets of \mathbb{C}^* to a holomorphic function with a zero at ∞ and at most a simple pole at 0. (The fact that the limit is holomorphic follows from (42).) Hence

$$\lim \widetilde{u}_{n,z} = \frac{c}{2z}$$

for some constant c . Recalling that $(\log|z|)_z = \frac{1}{2z}$, this gives (32) of Proposition 10. It remains to prove that $c > 0$. Let $\widehat{\phi}_n$ be the vertical flux on the closed curve of \widehat{M}_n that is the graph of \widehat{f}_n over the circle $C(0, 1) \subset \mathbb{C}^*$. Then by the same computation as at the end of Section 3.2,

$$\lim \frac{|\log t_n|}{\widehat{t}_n} \widehat{\phi}_n = 2\pi c.$$

Now by scaling and homology invariance of the flux, $\widehat{\phi}_n = \frac{\phi_{1,n}}{\mu_n}$, where $\phi_{1,n}$ is the vertical flux on the closed curve of M_n that is the graph of f_n over the circle $C(i, \varepsilon)$. Hence $c = c_1$ and c_1 is positive by Proposition 3. \square

5.5. Proof of Proposition 12 (Case 3b). Recall that in Case 3b, $\mu_n = |p_{1,n} - i|$ and for all $\beta > 0$, $\mu_n \geq t_n^\beta$ for n large enough. Let U_n be the domain defined in (38). Since $\mu_n \gg t_n^{1/8} \geq r'_{j,n}$ by (37), we have

$$\lim \widehat{U}_n = \mathbb{C} \setminus \{\widehat{p}_1, \dots, \widehat{p}_m\}.$$

(Compare with Case 3a, where the limit is \mathbb{C}^* .) Define again

$$\widetilde{u}_n = \frac{|\log t_n|}{\widehat{t}_n} (\widehat{u}_n - \widehat{u}_n(z_0)).$$

By the same argument as in Section 5.4 we obtain that $\widetilde{u}_{n,z}$ converges on compact subsets of $\mathbb{C} \setminus \{\widehat{p}_1, \dots, \widehat{p}_m\}$ to a meromorphic function with at most simple poles at $\widehat{p}_1, \dots, \widehat{p}_m$ and a zero at ∞ , so

$$\lim \widetilde{u}_{n,z} = \sum_{i=1}^m \frac{\widehat{c}_i}{2(z - \widehat{p}_i)}.$$

It remains to prove that the numbers $\widehat{c}_1, \dots, \widehat{c}_m$ are positive. For $1 \leq i \leq m$, let $\widehat{\phi}_{i,n}$ be the vertical flux of \widehat{M}_n on the graph of \widehat{f}_n restricted to the circle $C(\widehat{p}_i, \varepsilon)$. Then by the computation at the end of Section 3.2, we have

$$\lim \frac{|\log t_n|}{\widehat{t}_n} \widehat{\phi}_{i,n} = 2\pi \widehat{c}_i.$$

We will prove that \widehat{c}_i is positive by estimating the vertical flux using the height estimate as in Section 3.4. Take $\beta = \frac{1}{18(N+2)}$ and let

$$B_n = \bigcup_{i=1}^N D(p_{i,n}, t_n^\beta).$$

By Lemma 6 with $\alpha = \frac{1}{2}$, we have for n large enough:

$$u_n \leq (N+2) \frac{\beta}{\alpha} = \frac{t_n}{9} \quad \text{in } D(i, \varepsilon) \setminus B_n.$$

(Lemma 6 gives us this estimate for $|z| \leq 1$. The result follows because u_n is symmetric with respect to the unit circle). Consequently, the level set $u_n = \frac{t_n}{8}$ is contained in B_n . By the hypothesis of Case 3b, for n large enough, $\mu_n \gg t_n^\beta$ so the disks $D(p_{i,n}, t_n^\beta)$ for $1 \leq i \leq m$ are disjoint. Hence B_n has at least m components. Let $\Gamma_{i,n}$ be the component of the level set $u_n = \frac{t_n}{8}$ which encloses the point $p_{i,n}$ and $D_{i,n}$ the disk bounded by $\Gamma_{i,n}$. Then $D_{i,n}$ contains no other point $p_{j,n}$ with $1 \leq j \leq m$, $j \neq i$. (It might contain points $p_{j,n}$ with $j > m$). The proof of Proposition 4 in Section 3.3 gives us a point $p_{j,n} \in D_{i,n}$ (with either $j = i$ or $j > m$ and $\widehat{p}_j = \widehat{p}_i$) such that

$$r_{j,n} \geq C \frac{t_n}{|\log t_n|}$$

for some positive constant C . Scaling by $1/\mu_n$, this implies that

$$\widehat{\phi}_{i,n} \geq 2\pi \frac{C}{2} \frac{\widehat{t}_n}{|\log t_n|}.$$

Hence $\widehat{c}_i > 0$. \square

5.6. Proof of Proposition 13 (Case 3b). Let $g_n = \widehat{u}_{n,z}$. We have to prove that the cross-product term in (33) can be neglected, namely:

$$\operatorname{Re} \int_{C(\widehat{p}_1, \varepsilon)} w_{n,z}(z) g_n(z) (1 - \mu_n^2 z^2) dz = o\left(\frac{\widehat{t}_n}{(\log t_n)^2}\right).$$

The proof of this fact is the same as the proof of Proposition 5 in Section 3.4, with the following modifications:

- $\arg z$ is replaced by the function w_n defined in (31), so its derivative $\frac{1}{2iz}$ is replaced by $w_{n,z}$.
- $1 - z^2$ is replaced by $1 - \mu_n^2 z^2$.
- t_n, u_n , etc... now have hats: $\widehat{t}_n, \widehat{u}_n$, etc...
- From

$$w_{n,z} = \frac{1}{4\pi i} \left(\frac{1}{\mu_n z + i} - \frac{i}{1 + i\mu_n z} \right)$$

we deduce that $|w_{n,z}|$ is bounded in $D(\widehat{p}_1, \varepsilon)$ and since $\widehat{p}_{j,n} \in i\mathbb{R}$, that $w_{n,z}(\widehat{p}_{j,n})$ is real, which is what we need to ensure that the term $a_{j,1}$ does not contribute to the integral (see (30)). \square

APPENDIX A. AUXILIARY RESULTS

This appendix contains several results about minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ that have been used in the proof of Theorem 4. Some of these results are true for minimal surfaces in the Riemannian product $M \times \mathbb{R}$ where (M, g) is a 2-dimensional Riemannian manifold. These results are local, so we can assume without loss of generality that M is a domain $\Omega \subset \mathbb{C}$ equipped with a conformal metric $g = \lambda^2 |dz|^2$, where λ is a smooth positive function on Ω . Given a function f on Ω , the graph of f is a minimal surface in $M \times \mathbb{R}$ if it satisfies the minimal surface equation

$$(43) \quad \operatorname{div}_g \frac{\nabla_g f}{W} = 0 \quad \text{with } W = \sqrt{1 + \|\nabla_g f\|_g^2}$$

where the subscript g means that the quantity is computed with respect to the metric g , so for instance

$$\nabla_g f = \lambda^{-2} \nabla f, \quad \operatorname{div}_g X = \lambda^{-2} \operatorname{div}(\lambda^2 X).$$

In coordinates, (43) gives the equation

$$(44) \quad (1 + \lambda^{-2} f_y^2) f_{xx} + (1 + \lambda^{-2} f_x^2) f_{yy} - 2\lambda^{-2} f_x f_y f_{xy} + (f_x^2 + f_y^2) \left(\frac{\lambda_x}{\lambda} f_x + \frac{\lambda_y}{\lambda} f_y \right) = 0.$$

Propositions 15, 18, 19 and 20 will be formulated in this setup.

A.1. Interior gradient and Laplacian estimate.

Proposition 15. *Let Ω be a domain in \mathbb{C} equipped with a smooth conformal metric $g = \lambda^2 |dz|^2$. Let $f : \Omega \rightarrow \mathbb{R}$ be a solution of the minimal surface equation (43). Assume that $|f| \leq t$ in Ω and $\|\nabla f\| \leq 1$. Then*

$$\|\nabla f(z)\| \leq \frac{Ct}{d(z)}$$

$$|\Delta f(z)| \leq \frac{Ct^3}{d(z)^4}$$

for all $z \in \Omega$ such that $d(z) \geq t$. Here, $d(z)$ denotes the euclidean distance to the boundary of Ω . The gradient and Laplacian are for the euclidean metric. The constant C only depends on the diameter of Ω and on a bound on λ , λ^{-1} and its partial derivatives of first and second order.

Proof. Let us write the minimal surface equation (44) as $L(f) = 0$, where L is a second order linear elliptic operator whose coefficients depend on f_x and f_y . Theorem 12.4 in Gilbarg-Trudinger gives us a uniform constant C and $\alpha > 0$ such that (with Gilbarg-Trudinger notation)

$$[Df]_\alpha^{(1)} \leq C\|f\|_0 \leq Ct.$$

If $d(z, \partial\Omega) \geq t$, this implies

$$[Df]_\alpha^{(0)} \leq \frac{Ct}{t} = C.$$

Then we have the required C^α estimates of the coefficients of L to apply the interior Schauder estimate (Theorem 6.2 in Gilbarg-Trudinger):

$$|D^k f(z)| \leq \frac{C}{d(z)^k} \|f\|_0 \leq C \frac{t}{d(z)^k}, \quad k = 0, 1, 2.$$

The minimal surface equation (44) implies

$$|\Delta f| \leq C(|Df|^2 |D^2 f| + |Df|^3) \leq C \frac{t^3}{d^4}.$$

□

A.2. Alexandrov moving planes. We may use the Alexandrov reflection technique in $\mathbb{S}^2 \times \mathbb{R}$ with the role of horizontal planes played by the level spheres $\mathbb{S}^2 \times \{t\}$, and the role of vertical planes played by a family of totally geodesic cylinders. Specifically, let $E \subset \mathbb{S}^2 \times \{0\}$ be the closed geodesic that is the equator with respect to the antipodal points O , O^* , let $X \subset \mathbb{S}^2 \times \{0\}$ be a geodesic passing through O and O^* , and define E_θ to be the rotation of $E = E_0$ through an angle θ around the poles $E \cap X$. The family of geodesic cylinders

$$E_\theta \times \mathbb{R}, \quad -\pi/2 \leq \theta < \pi/2,$$

when restricted to the complement of $(E \cap X) \times \mathbb{R}$ is a foliation.

Proposition 16. *Let $\Gamma = \gamma_1 \cup \gamma_2$ with each γ_i a C^2 Jordan curve in $\mathbb{S}^2 \times \{t_i\}$, $t_1 \neq t_2$, that is invariant under reflection in $\Pi = E \times \mathbb{R}$. Suppose further that each component of $\gamma_i \setminus \Pi$ is a graph over Π with locally bounded slope. Then any minimal surface Σ with $\partial\Sigma = \Gamma$ that is disjoint from at least one of the vertical cylinders $E_\theta \times \mathbb{R}$, must be symmetric with respect to reflection in Π , and each component of $\Sigma \setminus \Pi$ is a graph of locally bounded slope over a domain in Π .*

(Given a domain $\mathcal{O} \subset \Pi$ and a function $f : \mathcal{O} \rightarrow [-\pi/2, \pi/2)$, the graph of f is the set of points $\{\text{rot}_{f(p)} p : p \in \mathcal{O}\}$, where rot_θ is the rotational symmetry that takes Π to $E_\theta \times \mathbb{R}$.)

The proof is the same as the classical proof for minimal surfaces in \mathbb{R}^3 using the maximum principle. (See for example Schoen [5] Corollary 2.)

A.3. Flux. Let N be a Riemannian manifold, $M \subset N$ a minimal surface and χ a Killing field on N . Let γ be a closed curve on M and μ be the conormal along γ . Define

$$\text{Flux}_\chi(\gamma) = \int_\gamma \langle \mu, \chi \rangle ds.$$

It is well known that this only depends on the homology class of γ .

Proposition 17. *In the case $N = \mathbb{S}^2(R) \times \mathbb{R}$, the space of Killing fields is 4 dimensional. It is generated by the vertical unit vector ξ , and the following three horizontal vectors fields:*

$$\chi_X(z) = \frac{1}{2} \left(1 + \frac{z^2}{R^2} \right)$$

$$\chi_Y(z) = \frac{i}{2} \left(1 - \frac{z^2}{R^2} \right)$$

$$\chi_E(z) = \frac{iz}{R}$$

These vector fields are respectively unitary tangent to the great circles X , Y and E . They are generated by the one-parameter families of rotations about the poles whose equators are these great circles.

Proof: The isometry group of $\mathbb{S}^2(R) \times \mathbb{R}$ is well known to be 4-dimensional. Recall that our model of $\mathbb{S}^2(R)$ is $\mathbb{C} \cup \infty$ with the conformal metric $\frac{2R^2}{R^2 + |z|^2} |dz|$. By differentiating the 1-parameter group $z \mapsto e^{it}z$ of isometries of \mathbb{S}^2 , we obtain the horizontal Killing field $\chi(z) = iz$, which suitably normalized gives χ_E . Let

$$\varphi(z) = \frac{Rz + iR^2}{iz + R}.$$

This corresponds, in our model of $\mathbb{S}^2(R)$, to the rotation about the x -axis of angle $\pi/2$. It maps the great circle E to the great circle X . We transport χ_E by this isometry to get the Killing field χ_X : a short computation gives

$$\chi_X(z) = \varphi_* \chi_E(z) = \varphi'(\varphi^{-1}(z)) \chi_E(\varphi^{-1}(z)) = \frac{z^2 + R^2}{2R^2}.$$

Then we transport χ_X by the rotation $\psi(z) = iz$ to get the Killing field χ_Y :

$$\chi_Y(z) = \psi_* \chi_X(z) = i \frac{(-iz)^2 + R^2}{2R^2}.$$

□

Proposition 18. *Let $\Omega \subset \mathbb{C}$ be a domain equipped with a conformal metric $g = \lambda^2 |dz|^2$. Let $f : \Omega \rightarrow \mathbb{R}$ be a solution of the minimal surface equation (43). Let γ be a closed, oriented curve in Ω and ν be the euclidean exterior normal vector along γ (meaning that (γ', ν) is a negative orthonormal basis). Let M be the graph of f and let $\tilde{\gamma}$ be the closed curve in M that is the graph of f over γ .*

(1) *For the vertical unit vector ξ ,*

$$\text{Flux}_\xi(\tilde{\gamma}) = \int_\gamma \frac{\langle \nabla f, \nu \rangle}{W}$$

where W is defined in equation (43). (Here the gradient, scalar product and line element are euclidean.) If $\|\nabla f\|$ is small, this gives

$$\text{Flux}_\xi(\tilde{\gamma}) = \text{Im} \int_\gamma (2f_z + O(|f_z|^2)) dz$$

(2) If χ is a horizontal Killing field,

$$\text{Flux}_\chi(\tilde{\gamma}) = -\text{Im} \int_\gamma (2(f_z)^2 \chi(z) + O(|f_z|^4)) dz.$$

Proof: Let (N, g) be the Riemannian manifold $\Omega \times \mathbb{R}$ equipped with the product metric $g = \lambda^2 |dz|^2 + dt^2$. Let M be the graph of f , parametrized by

$$\psi(x, y) = (x, y, f(x, y)).$$

The unit normal vector to M is

$$n = \frac{1}{W} (-\lambda^{-2} f_x, -\lambda^{-2} f_y, 1).$$

Assume that γ is given by some parametrization $t \mapsto \gamma(t)$, fix some time t and let $(X, Y) = \gamma'(t)$. Then

$$d\psi(\gamma') = (X, Y, Xf_x + Yf_y)$$

is tangent to $\psi(\gamma)$ and its norm is ds , the line element on M . We need to compute the conormal vector in N . The linear map $\varphi : (T_p N, g) \rightarrow (\mathbb{R}^3, \text{euclidean})$ defined by

$$\varphi(u_1, u_2, u_3) = (\lambda u_1, \lambda u_2, u_3)$$

is an isometry. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be two orthogonal vectors in $T_p N$. Let

$$w = \varphi^{-1}(\varphi(u) \wedge \varphi(v)) = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ \lambda^2(u_1 v_2 - u_2 v_1) \end{pmatrix}.$$

Then (u, v, w) is a direct orthogonal basis of $T_p N$ and $\|w\| = \|u\| \|v\|$. We use this with $u = d\psi(\gamma')$, $v = n$. Then $w = \mu ds$, where μ is the conormal to $\psi(\gamma')$. This gives

$$\mu ds = \frac{1}{W} \begin{pmatrix} Y + \lambda^{-2} f_y (Xf_x + Yf_y) \\ -X - \lambda^{-2} f_x (Xf_x + Yf_y) \\ -f_y X + f_x Y \end{pmatrix}.$$

For the vertical unit vector $\xi = (0, 0, 1)$, this gives

$$\text{Flux}_\xi(\tilde{\gamma}) = \int_\gamma \frac{-f_y dx + f_x dy}{W} = \int_\gamma \frac{\langle \nabla f, \nu \rangle}{W}.$$

The second formula of point (1) follows from $W = 1 + O(\|\nabla f\|^2)$ and

$$\text{Im}(2f_z dz) = \text{Im}((f_x - if_y)(dx + idy)) = f_x dy - f_y dx.$$

To prove point (2), let χ be a horizontal Killing field, seen as a complex number. Then

$$\langle \chi, \mu ds \rangle_g = \lambda^2 \text{Re} \left(\frac{\chi}{W} (Y + iX + \lambda^{-2}(f_y + if_x)(Xf_x + Yf_y)) \right)$$

Hence

$$\text{Flux}_\chi(\tilde{\gamma}) = \text{Re} \int_\gamma \frac{\lambda^2 \chi}{W} (dy + i dx) + \frac{\chi}{W} (f_y + if_x)(f_x dx + f_y dy).$$

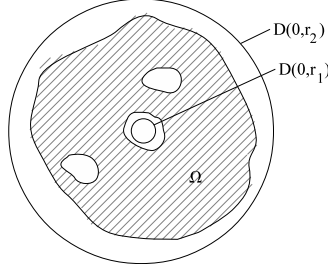


FIGURE 3.

We then expand $1/W$ as a series

$$\frac{1}{W} = 1 - \frac{1}{2}\lambda^{-2}(f_x^2 + f_y^2) + O(|\nabla f|^4).$$

This gives after some simplifications

$$\text{Flux}_\chi(\tilde{\gamma}) = \text{Re} \int_\gamma \lambda^2 \chi(dy + i dx) + \text{Re} \int_\gamma \frac{i}{2} \chi(f_x - i f_y)^2(dx + i dy) + O(|\nabla f|^4).$$

The second term is what we want. The first term, which does not depend on f , vanishes. Indeed, if $f \equiv 0$ then M is $\Omega \times \{0\}$ and the flux we are computing is zero (by homology invariance of the flux, say). \square

A.4. Height estimate. The following proposition tells us that a minimal graph with small vertical flux cannot climb very high. It is the key to estimate from below the size of the catenoidal necks.

Proposition 19. *Let $\Omega \subset \mathbb{C}$ be a domain that consists of a (topological) disk D minus $n \geq 1$ topological disks D_1, \dots, D_n contained in D . We denote by Γ the boundary of D and by γ_i the boundary of D_i . Assume that D_1 contains $D(0, r_1)$ and D is contained in $D(0, r_2)$, for some numbers $0 < r_1 < r_2$. (Here r_1, r_2 are euclidean lengths). (See Figure 3).*

Assume that Ω is equipped with a conformal metric $g = \lambda^2 |dz|^2$. Let $f : \Omega \rightarrow \mathbb{R}$ be a solution of the minimal surface equation (43). Assume that

- (1) $f \equiv 0$ on Γ .
- (2) $f \equiv -h < 0$ is constant on γ_1 .
- (3) f is constant on γ_i for $2 \leq i \leq n$, with $-2h \leq f \leq 0$.
- (4) $\partial f / \partial \nu \leq 0$ on γ_i for $1 \leq i \leq n$.
- (5) $\|\nabla_g f\|_g \leq 1$ in Ω

Let ϕ be the vertical flux on Γ :

$$\phi = \int_\Gamma \frac{\langle \nabla f, \nu \rangle}{W} > 0$$

Then

$$h \leq \frac{\sqrt{2}}{\pi} \phi \log \frac{r_2}{r_1}.$$

(Note that Hypothesis (4) is always satisfied if $f \equiv -h$ on all γ_i by the maximum principle.)

Proof. Let A be the annulus $D(0, r_2) \setminus D(0, r_1)$. Write $|df|$ for the euclidean norm of the euclidean gradient of f . Let ρ be the function equal to $|df|$ on Ω and 0 on $\mathbb{C} \setminus \Omega$. Then

$$\begin{aligned}
\iint_A \rho^2 dx dy &= \iint_\Omega \|\nabla_g f\|_g^2 d\mu_g \quad \text{by conformal invariance of the energy} \\
&\leq \sqrt{2} \iint_\Omega \left\langle \frac{\nabla_g f}{W}, \nabla_g f \right\rangle_g d\mu_g \quad \text{because } W \leq \sqrt{2} \text{ by point (5)} \\
&= \sqrt{2} \iint_\Omega \operatorname{div}_g \left(f \frac{\nabla_g f}{W} \right) d\mu_g \quad \text{by the minimal surface equation (43)} \\
&= \sqrt{2} \int_{\partial\Omega} \frac{f}{W} \langle \nabla_g f, \nu_g \rangle_g ds_g \quad \text{by the divergence theorem} \\
&= \sqrt{2} \int_{\partial\Omega} \frac{f}{W} \langle \nabla f, \nu \rangle \quad \text{where now all quantities are euclidean} \\
&= \sqrt{2} \sum_{i=1}^n \int_{\gamma_i} \frac{f}{W} \langle \nabla f, \nu \rangle \quad \text{by point (1)} \\
&\leq -2\sqrt{2}h \sum_{i=1}^n \int_{\gamma_i} \frac{\langle \nabla f, \nu \rangle}{W} \quad \text{by points (3) and (4)}
\end{aligned}$$

Hence by homology invariance of the flux,

$$(45) \quad \iint_A \rho^2 dx dy \leq 2\sqrt{2}h\phi$$

Consider the ray from $r_1 e^{i\theta}$ to $r_2 e^{i\theta}$. The integral of df along this ray, intersected with Ω , is equal to h . (If the ray happens to enter one of the disks D_i , then this is true because f is constant on ∂D_i .) Integrating for $\theta \in [0, 2\pi]$ we get

$$\begin{aligned}
2\pi h &\leq \int_{r=r_1}^{r_2} \int_{\theta=0}^{2\pi} \rho dr d\theta \\
&= \int_A \frac{\rho}{r} dx dy \\
&\leq \left(\iint_A \rho^2 dx dy \right)^{1/2} \left(\iint_A \frac{1}{r^2} dx dy \right)^{1/2} \quad \text{by Cauchy Schwartz} \\
&\leq \left(2\sqrt{2}h\phi \right)^{1/2} \left(2\pi \log \frac{r_2}{r_1} \right)^{1/2} \quad \text{using (45)}
\end{aligned}$$

The proposition follows. \square

The next proposition is useful to find circles on which we have a good estimate of $\int |df|$.

Proposition 20. *Under the same hypotheses as Proposition 19, consider some point $p \in \Omega$. Given $0 < r'_1 < r'_2$, there exists $r \in [r'_1, r'_2]$ such that*

$$\int_{C(p,r) \cap \Omega} |df| \leq \sqrt{8}\phi \left(\log \frac{r_2}{r_1} \right)^{1/2} \left(\log \frac{r'_2}{r'_1} \right)^{-1/2}.$$

Proof: Consider the function

$$F(r) = \int_{C(p,r) \cap \Omega} |df| = \int_{\theta=0}^{2\pi} \rho(p + re^{i\theta}) r d\theta.$$

Then

$$\begin{aligned}
& \min_{r'_1 \leq r \leq r'_2} F(r) \log \frac{r'_2}{r'_1} \\
& \leq \int_{r=r'_1}^{r'_2} \frac{F(r)}{r} dr \\
& = \int_{r=r'_1}^{r'_2} \int_{\theta=0}^{2\pi} \frac{\rho(p + re^{i\theta})}{r} r dr d\theta \\
& \leq \left(\int_{r'_1}^{r'_2} \int_0^{2\pi} \rho(p + re^{i\theta})^2 r dr d\theta \right)^{1/2} \left(\int_{r'_1}^{r'_2} \int_0^{2\pi} \frac{1}{r^2} r dr d\theta \right)^{1/2} \\
& \leq \left(\int_A \rho^2 dx dy \right)^{1/2} \left(2\pi \log \frac{r'_2}{r'_1} \right)^{1/2} \\
& \leq \left(8\phi^2 \log \frac{r_2}{r_1} \log \frac{r'_2}{r'_1} \right)^{1/2} \quad \text{using (45) and Proposition 19}
\end{aligned}$$

The proposition follows. \square

A.5. A Laurent-type formula for C^1 functions.

Proposition 21. *Let $\Omega \subset \mathbb{C}$ be a domain of the form*

$$\Omega = D(0, R) \setminus \bigcup_{i=1}^n \overline{D}(p_i, r_i).$$

Here we assume that the closed disks $\overline{D}(p_i, r_i)$ are disjoint and are included in $D(0, R)$. Let f be a C^1 function on $\overline{\Omega}$. Then in Ω ,

$$f(z) = f^+(z) + \sum_{i=1}^n f_i^-(z) + \frac{1}{2\pi i} \int_{\Omega} \frac{f_{\bar{z}}(w)}{w - z} dw \wedge \overline{dw}$$

where f^+ is holomorphic in $D(0, R)$ and each f_i^- is holomorphic in $\mathbb{C} \setminus \overline{D}(p_i, r_i)$. Moreover, these functions have the following series expansion

$$\begin{aligned}
f^+(z) &= \sum_{k=0}^{\infty} a_k z^k \quad \text{with } a_k = \frac{1}{2\pi i} \int_{C(0, R)} \frac{f(z)}{z^{k+1}} dz \\
f_i^-(z) &= \sum_{k=1}^{\infty} \frac{a_{i,k}}{(z - p_i)^k} \quad \text{with } a_{i,k} = \frac{1}{2\pi i} \int_{C(p_i, r_i)} f(z) (z - p_i)^{k-1} dz
\end{aligned}$$

The series converge uniformly in compact subsets of Ω .

Remark 11. *This is the same as the Laurent series theorem except that there is a correction term which vanishes when f is holomorphic. The integration circles in the formula for a_n and $a_{i,n}$ cannot be changed (as in the classical Laurent series theorem) since f is not holomorphic.*

Proof. By Cauchy Pompeiu integral formula for C^1 functions:

$$(46) \quad f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(w)}{w - z} dw + \frac{1}{2\pi i} \int_{\Omega} \frac{f_{\bar{z}}(w)}{w - z} dw \wedge \overline{dw}.$$

Define

$$f^+(z) = \frac{1}{2\pi i} \int_{C(0,R)} \frac{f(w)}{w-z} dw$$

$$f_i^-(z) = -\frac{1}{2\pi i} \int_{C(p_i, r_i)} \frac{f(w)}{w-z} dw$$

The function f^+ is holomorphic in $D(0, R)$. The function f_i^- is holomorphic in $\mathbb{C} \setminus D(p_i, r_i)$ and extends at ∞ with $f_i^-(\infty) = 0$. These two functions are expanded in power series exactly as in the proof of the classical theorem on Laurent series (see e.g. Conway [3] page 107). \square

Proposition 22. *Let $\Omega \subset \mathbb{C}$ be a domain as in Proposition 21. Let $u : \Omega \rightarrow \mathbb{R}$ be a real-valued function of class C^2 . Take $f = \partial u / \partial z$. Then the coefficients $a_{i,1}$ which appear in the conclusion of Proposition 21 are real.*

Proof.

$$\begin{aligned} \operatorname{Im} a_{i,1} &= \frac{-1}{2\pi} \operatorname{Re} \int_{C(p_i, r_i)} u_z dz \\ &= \frac{-1}{4\pi} \int_{C(p_i, r_i)} u_z dz + u_{\bar{z}} d\bar{z} \quad \text{because } u \text{ is real valued} \\ &= \frac{-1}{4\pi} \int_{C(p_i, r_i)} du = 0 \quad \text{because } u \text{ is well defined in } \Omega \end{aligned}$$

\square

A.6. Residue computation.

Proposition 23.

$$\operatorname{Res}_p (\log z - \log p)^{-1} = p, \quad \operatorname{Res}_p \left(\frac{1-z^2}{4z^2} \right) (\log z - \log p)^{-2} = -\frac{1+p^2}{4p}.$$

Proof.

$$\log z - \log p = \log \left(1 + \frac{z-p}{p} \right) = \frac{z-p}{p} - \frac{1}{2} \left(\frac{z-p}{p} \right)^2 + O((z-p)^3)$$

The first residue follows. Then

$$(\log z - \log p)^{-2} = \left(\frac{z-p}{p} \right)^{-2} \left(1 - \frac{1}{2} \left(\frac{z-p}{p} \right) \right)^{-2} = \frac{p^2}{(z-p)^2} + \frac{p}{z-p} + O(1).$$

Let

$$f(z) = \frac{1-z^2}{4z^2} = \frac{1}{4z^2} - \frac{1}{4}.$$

Then

$$\begin{aligned} \operatorname{Res}_p \left(\frac{1-z^2}{4z^2} (\log z - \log p)^{-2} \right) &= \operatorname{Res}_p \left(\frac{f(z)p^2}{(z-p)^2} \right) + \operatorname{Res}_p \left(\frac{f(z)p}{(z-p)} \right) \\ &= f'(p)p^2 + f(p)p \quad (\text{by the Taylor expansion for } f \text{ at } p) \\ &= -\frac{1}{2p^3}p^2 + \frac{1-p^2}{4p} \\ &= -\frac{1+p^2}{4p}. \end{aligned}$$

\square

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